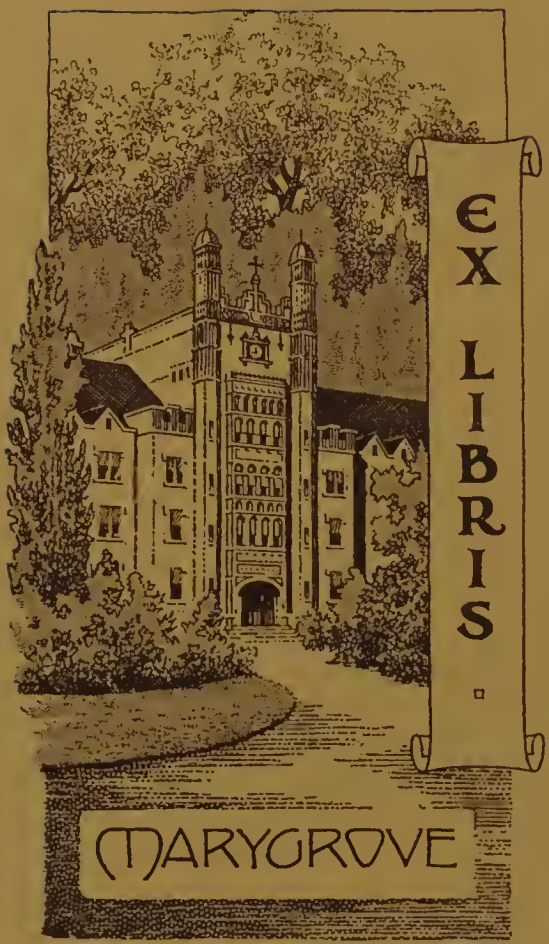
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PLANE ANALYTIC GEOMETRY

PLANE ANALYTIC GEOMETRY

BY

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PLANE ANALYTIC GEOMETRY

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EDITOR'S PREFACE

I believe it may be truly said that the three preceding texts of this series have established a definite standard of excellence in a number of important respects which are enumerated in the Author's Preface of the present book. If any of these features should be singled out for special mention by the editor, he would select those which contribute most strongly to the two phases of pupil-welfare that are uppermost today in the minds of all progressive teachers in both secondary schools and colleges: namely, (1) the welfare of the average pupil and (2) the welfare of the exceptionally gifted pupil.

A careful analysis of the outstanding features of the three preceding books of this series, and likewise of the present text, will reveal the fact that the average pupil is well provided for by the clarity and simplicity of statement throughout, by the unusually careful gradation of exercises, and by the cumulative reviews which are built on the psychological doctrine of "remembering and forgetting." On the other hand, the exceptional pupil is equally well provided for by groups of exercises and problems which challenge his mettle and his initiative, and by certain whole chapters of subject matter which serve to enlarge his horizon, stimulate his imagination, and lead him on to a new appreciation of mathematics.

The leaders of tomorrow in the mathematical world must come from the small group of exceptional pupils in our classes of today. Hence, any pedagogical procedure that best promotes the scientific stimulation of this group, while by no means overlooking the welfare of the larger group of average pupils, is to be commended and encouraged. Possibly the chief merit of these texts lies along this line.

Again I must emphasize the importance of the under-graduate

mathematical club as an auxiliary to the classroom, not only for the opportunity which it affords to the exceptional pupil but also for the stimulus thus provided for all students majoring in mathematics. The importance of this idea has taken on such great significance in the past few years that the undergraduate club is fast becoming an institution which serves as a good barometer for indicating the progressive character of any collegiate mathematical department.

H. E. Slaught.

The University of Chicago
September, 1929

AUTHORS' PREFACE

This is the fourth volume in the Harper series of college texts on mathematics, under the general editorship of Professor H. E. Slaught. The present book is a collaboration by the authors of *Plane Trigonometry;* and the remaining two, the *College Algebra* and *A Survey Course in Mathematics,* are by N. J. Lennes as sole author. The work of organizing such a series of texts and bringing them up to a fairly uniform standard of excellence is a task of considerable magnitude. This task was undertaken with the purpose of embodying in these books certain characteristics and features, among which the more important are: flexibility, clarity and simplicity of statement, convenient and attractive arrangement and paging, exceptionally well-graded exercises and problems, provisions for extra work by stronger students, means for adequate reviews, and a final chapter which will enable the student to learn something of the historical setting of the subject.

Naturally there must be some repetition in the successive prefaces of these books. We paraphrase in part the preface to the *College Algebra:*

Flexibility of the book.—The book is designed to be readily usable by those who do not feel the need for some of its special features or who may even disapprove sharply of plans for which it provides; it is so arranged that those who desire no innovations may use it without being disturbed in the slightest. Our object has been to arrange these features so they will not obtrude themselves upon the user, while being available for those who may wish to use them. Should not the writers of a text seek to provide an excellent teaching instrument, which the individual teacher may use to carry out his own plans in his own way, rather than to produce an inflexible book, which, if

used at all, would make it difficult to depart very far from the authors' special aims and ways of attaining them? Even a very slight sense of humor and the most meager acquaintance with the facts that are pertinent make the answer obvious.

Provisions for the abler student.—The most pressing of the special needs we have come to feel is an arrangement making it easy and convenient to give the abler student the kind of work he should have. In a normal class of, say, twenty-five there is a large range of ability. Some are slow and incapable, and it is only with the greatest effort that they can do the work well enough to obtain a passing grade, while a few can do it with great ease, and could (and would readily) do work of considerably greater difficulty if the occasion required. No doubt the majority of teachers more or less consciously pitch their work to suit fairly nearly the median ability of the class. The main body of this text is purposed to suit such ability. The plan provided for here, and in use for many years in our own classes, is to give extra work of a somewhat more difficult nature which the abler students may undertake *if they desire to do so.* In our own practice such extra work is usually not considered in the classroom. Reports on it are made privately to the instructor during office hours. The time required on the part of the instructor is not great, since the number who undertake such work is small—five or six in a class of twenty-five is a normal number. Again, the undergraduate mathematics club serves as an outlet. A freshman may present a half-hour paper to this club on some higher plane curves not studied in the class, or on diameters, or on poles and polars, or on some other available topic. The very able freshman does this as well as the ordinary senior, and the interest and stimulus he derives from it are greater than might come from several courses of ordinary class work. In our practice a grade of A is practically never given except to those who, on their own initiative, have undertaken and successfully carried through such a piece of work.

In our own institution Chapters I to VIII inclusive and parts

of Chapters IX, XIII, XIV, together with the cumulative reviews in Chapter XVI, form the main body or "minimum core" of the course in analytical geometry. From this are usually excluded the sections entitled "Solution of Problems" in Chapters VI, VII, VIII, and parts or all of the Miscellaneous Problems at the ends of these chapters.

This leaves available for special work all of Chapters XI, XII, XV, and the omitted parts of other chapters. The special contacts between the instructor and the abler student encouraged by this plan are of great value. In many cases these contacts and the special work undertaken on the student's own initiative become landmarks in his college career. We have found no plan so effective as this one in arousing able students to aggressive work and in stimulating them to self-direction.

The cumulative reviews.—Much has been said and written in deprecation of "mere memory work." But what is deprecated is not the remembering, but memorizing without understanding. Obviously, remembering is not only desirable but necessary. It is precisely in this field of memory that the investigators in psychology during the last fifty years have obtained some of their most significant and best substantiated results. Vivid initial impressions, and understanding of the meaning of that which is learned, shorten the time of learning and increase retention; this is a matter of common sense (and is confirmed by laboratory experiments), and account of it is no doubt taken very generally in teaching.

But there is in this connection another fact which, we believe, is not taken into account very generally; certainly our texts make no provision for taking account of it. We are always forgetting; and to secure relatively permanent retention, a *series* of relearnings is necessary. This has been confirmed by the results of experiments without number. We study a topic and make it fairly clear to ourselves; but in a crowded life it is laid aside and in a few days is forgotten, so that without prompting

we cannot recall any of it. Relearning, however, is much easier than the first learning. Each successive restudy becomes easier than the preceding, and after a series of such relearnings, retention becomes relatively permanent. No fact in connection with learning is more important than this, and no fact has been neglected more consistently. The teacher, who has relearned his subject so many times, finds it hard to understand how naturally and completely the student forgets. The usual procedure is a review (a single relearning) just before the final examination, and after that undisturbed oblivion.

The cumulative reviews in Chapter XVI of this book make provision for systematic relearnings. There are forty-eight groups of exercises, six in each group, and the plan is to assign these groups in order, one each day. The subject matter of the text has been analyzed and the exercises in the cumulative reviews are so spaced that each topic recurs approximately with the proper spacing and frequency to conform to the "curve of forgetting and relearning." On the whole, the review exercises are rather simple, but at the same time they contain such elements of novelty as to provoke thoughtful relearning.

However, the main body of this text is exactly as it would be if these reviews were not in the book. In our own practice the study of the course proceeds as if they were not in existence, and they are assigned each day as a sort of side issue; but they do make for a type of retention which without them is simply impossible. For one thing, they make a review of the usual type unnecessary. Instead, a final review may become a consideration of relations between parts now well known and of the general bearing and significance of the whole subject, such as is provided for in Chapter XVII of this book. It is our belief that the systematic use of such reviews is the single greatest general improvement that is immediately and easily possible in our teaching, and that the neglect of them is one of the two

greatest pedagogical sins committed. The other is the neglect of the gifted student.

The historical sketch.—A historical sketch is given at the end of the book. Rarely does a student go to the treatises for such material, and even when he does he finds it no small task to pick out the account of his particular subject. If the story of a subject such as analytic geometry is rightly read, it is an interesting and illuminating chapter in the history of culture; it illustrates the fairly universal difficulty which man has had in making new discoveries and the frequent slowness with which discoveries are recognized after they are made. The plan of inserting "historical notes" in the text has often been used, but such notes cannot tell a connected story, and they seem to us decidedly inferior in value when compared with a single unified historical account.

Miscellaneous characteristics.—Unfortunately, there are some questions in connection with the organization of a course in analytic geometry on which there is divergence of opinion. On some of these the authors are frankly in doubt; and we can only hope that reviewers and critics may not be too certain that we are wrong in the course that has been followed. Shall the conic sections be studied separately first and brought together later under the general topic of the second degree equation, or shall the order be essentially reversed? The former of these two possibilities has been chosen, and we believe we have fairly conclusive reasons for so doing, though we are well aware that unanimous agreement cannot be secured. This is no doubt the most important point of disagreement.

In dealing with the normal equation of the straight line, shall the distance from the origin to the line be regarded as always positive? We have answered this question in the affirmative, although more than one excellent text adopts the contrary course. The reasons for our treatment are stated in Chapter XVII of the text.

Those equations which embody the main results of our study are printed in boldfaced type and are specially numbered. Do these represent a too minute study of the conic sections? We are not certain on this point, but we do not think so.

A general principle which has guided us is that the subject matter of a first course in analytic geometry must of necessity be the straight line and curves of the second degree, and that in such a course it is best to proceed from the particular to the general.

No opportunity has been neglected to give the student a look behind the scenes. At the beginning of each chapter the topic to be treated is stated briefly, and something about its relations to other topics is brought out.

More than usual care has been taken with the general appearance of the book; the work of paging alone has been no little task, and it is hoped that the users of the book will find it more than ordinarily convenient in many ways.

For convenient reference a short chapter has been added containing the main results from algebra and geometry of which use is made in the text. A brief table of powers, roots, natural sines and cosines will facilitate some of the computation. A fairly full index will make subsequent reference easy and should enhance the value of the book as a permanent possession for reference.

Prepublication use of this book.—In connection with the final work on the page proof, a class of twenty regular students used the proof-sheets as a text, as did also a smaller class of special and more mature students; and the book owes much to their frank and helpful criticism. Every exercise and problem in the whole book was solved by some student in the regular class, though of course no one student solved them all. All chapters were either studied in class or were reported on by individual students.

As in the other books of this series, Professor Slaught, the editor, has rendered painstaking and efficient service. Two

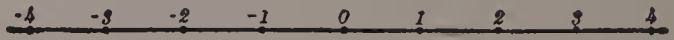
different drafts of the manuscript were read and criticised by him, as were also the galley- and the page-proofs. The book has had the full benefit of his large experience in teaching college mathematics.

N. J. Lennes
A. S. Merrill

PLANE ANALYTIC GEOMETRY

If a and b are any two real numbers, then $a + b$, $a - b$, ab, $\frac{a}{b}$, are always real numbers with the single exception that in $\frac{a}{b}$, b cannot be zero. Division by zero is impossible.

2. The geometric scale of real numbers.—On a straight line mark any point 0 (zero) and lay off consecutive segments of equal length in both directions from this point. Mark the division points in one direction (to the right) 1, 2, 3, . . . and the division points in the other direction -1, -2, -3, . . . The number marked 0 is called the **origin.**



The numbers marking these points give the distance and direction of each point from the origin. By using fractions and other real numbers a number may be assigned to every point on the line. The line marked in this way is called the **scale of real numbers.**

3. Directed segments.—In analytic geometry one of the end-points of a segment is regarded as the **beginning** or **initial point** of the segment and the other as the **terminal point.** In naming a segment the point given first is the initial point. Thus, in the segment AB, A is the initial point, while B is the initial point if the segment is given as BA.

If on the real number scale the initial point is represented by a smaller number than the terminal point, then the segment is positive; otherwise it is negative. If x_1 is the number representing the initial point of a segment and x_2 the number representing the terminal point, then $x_2 - x_1$ is the **number value** of the segment.

Since $x_2 - x_1 = -(x_1 - x_2)$, it follows that $AB = -BA$.

On a horizontal line the scale is usually arranged so that a

segment which extends toward the right from its initial point is positive, and on a vertical line the scale is arranged so that a segment which extends upward from its initial point is positive. Segments extending in the opposite directions are negative.

4. Addition of directed segments.—It follows directly from $AB = -BA$ that

$$AB + BA = 0. \tag{1}$$

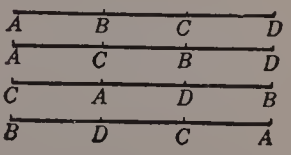
In this equation AB may be either positive or negative; in either case the sign of BA is the opposite of that of AB and equation (1) holds.

Directed segments may be used to represent many kinds of physical phenomena, such as motion, for instance. If the segments represent motion, then equation (1) simply says that moving from A to B and then from B to A brings one to the starting point.

It is evident that

$$AB + BC + CD = AD, \tag{2}$$

no matter in what order the points A, B, C, D are arranged on the line.



We also have

$$AB + BC + CA = 0. \tag{3}$$

SIGHT WORK

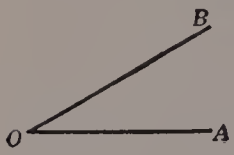
1. Find the number values of the segments whose end-points are: 1 and 7, − 2 and 6, 5 and 2, − 9 and − 3, 4 and − 2, − 3 and − 10, the initial point in each case being given first.

2. Indicate the sum $AB + BC + CD$ by using the numerical values of the segments (a) if A, B, C, D represent the numbers 2, 4, 6, 9; (b) if they represent 4, 9, 6, 10; (c) if they represent 6, −4, 5, −15; (d) if they represent 8, 0, 15, −4.

3. Verify that $AB + BC + CD + DA = 0$, using the number representations of the points given in (a),. . . (d) in Ex. 2.

4. If A, B, C, D represent the temperature readings 42°, 31°, 54°, 27°, verify that $AB + BC + CD = AD$.

5. Directed angles.—Angles may be regarded as signed or directed in exactly the same manner as the segments on a line. Either side of the angle may be regarded as its **initial side,** and the other as the **terminal side.**



If OA is the initial side of an angle and OB its terminal side, the angle is designated by $\angle AOB$; while if OB is the initial side and OA the terminal side, then the angle is designated by $\angle BOA$.

As in the case of segments,

$$\angle AOB = -\angle BOA.$$

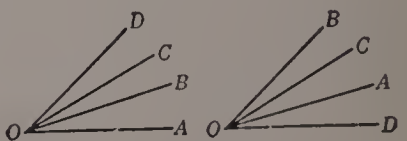
From this follow equations analogous to (1), (2), and (3) of § 4.

That is,

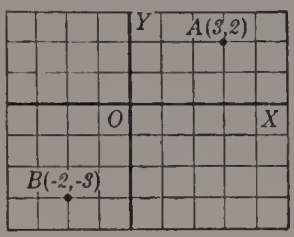
$$\angle AOB + \angle BOA = 0,$$

$$\angle AOB + \angle BOC + \angle COD = \angle AOD,$$

$$\angle AOB + \angle BOC + \angle COA = 0.$$



6. Rectangular coördinates.—If two lines are drawn at right angles to each other, then any point in the plane of these lines may be located by giving its distance and direction from each of the lines.

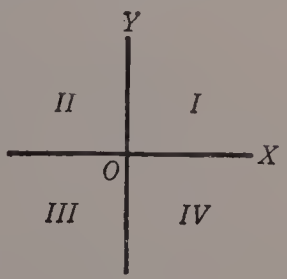


The two lines from which the distances are measured are called the **coördinate axes;** the horizontal line is usually called the ***x*-axis** and the vertical line the ***y*-axis.** The point in which these lines meet is called the **origin,** and is denoted by O. Distances to the right of the vertical line are regarded as positive, and distances to the left of it as negative; while distances upward from the horizontal line are positive, and distances downward are negative.

The distance and direction of a point from the y-axis (parallel to the x-axis) are given by what is called the **x-coördinate,** or **abscissa,** of the point, while the distance and direction from the x-axis (parallel to the y-axis) are given by what is called the **y-coördinate,** or **ordinate,** of the point.

The ordinate and abscissa of a point are together called its **coördinates.**

In giving the coördinates of a point, the x-coördinate is given first. Thus in the last figure on page 4, the point A is denoted by (3, 2) and not by (2, 3). Similarly the point B is denoted by $(-2, -3)$. A point A with coördinates (3, 2) is often denoted by $A(3, 2)$. In analytic geometry "finding a point" consists in finding its coördinates.



The four parts into which the coördinate axes divide the plane are called **quadrants.** The quadrants are numbered in order as shown in the figure.

Locating a point by means of coördinates is closely analogous to the well-known method of locating a point on the earth's surface by giving its latitude and longitude. Instead of the equator and the prime meridian we use the coördinate axes; and instead of east and west, north and south we use the plus and minus signs.

SIGHT WORK

1. In what quadrants are abscissas positive and in what quadrants are they negative?

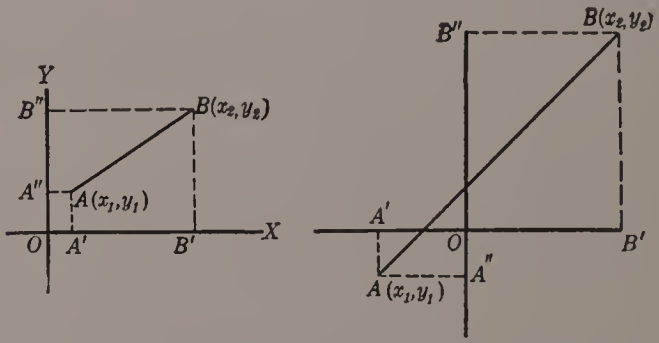
2. In what quadrants are ordinates positive and in what quadrants are they negative?

3. In what quadrants do the coördinates of a point have the same sign? In what quadrants are these signs opposite?

4. What is the locus of points whose abscissas are zero? of points whose ordinates are zero?

5. Comment on this statement: In analytic geometry every point in the plane is determined by a pair of real numbers; and conversely, every pair of real numbers determines a point.

7. Projection of a segment on the coördinate axes.—The length of the projection of a segment on the x-axis is equal numerically to the difference of the x-coördinates of its end-points. Likewise the length of the projection of a segment on the y-axis is numerically equal to the difference of the y-coördinates of the end-points.

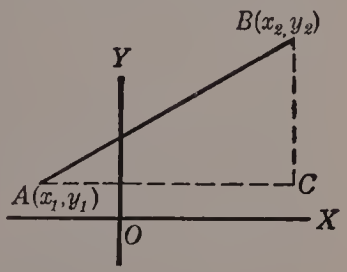


These statements are easily verified in the figures. In the first figure x_1 and x_2 are both positive, as are also y_1 and y_2. Clearly $A'B' = x_2 - x_1$ and $A''B'' = y_2 - y_1$. In the second figure x_1 and y_1 are both negative. In this case $A'B'$ is equal to the sum of the numerical values of x_1 and x_2, which is equal to $x_2 - x_1$, since x_1 is negative. Likewise, $A''B'' = y_2 - y_1$.

Note that the projection on the x-axis of a segment whose initial point is (x_1, y_1) and terminal point (x_2, y_2) is positive or negative according as $x_2 - x_1$ is positive or negative. A similar remark applies to the projection on the y-axis.

8. Projection of a segment on lines parallel to the coördinate axes.—The projections of a segment on two parallel lines are equal and hence it follows that the projection on a line parallel to the x-axis is equal to the difference between the x-coördinates of the end-points. Similarly, the projection of a segment on a line parallel to the y-axis is equal to the difference of the y-coördinates of its end-points. On the next page use is made of this fact.

9. The distance between two points.—To find the distance between $A(x_1, y_1)$ and $B(x_2, y_2)$, draw the lines AC and BC parallel to the two axes, forming the right triangle ABC. Then by §8, $AC = x_2 - x_1$, and $CB = y_2 - y_1$. Hence we have, putting $AB = d$,



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad [1]$$

Notice that we may equally well write

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

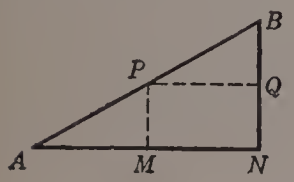
since $(x_1 - x_2)^2 = (x_2 - x_1)^2$ and $(y_1 - y_2)^2 = (y_2 - y_1)^2$.

Hence we may state the above formula in words as follows:

The distance between two points is found by adding the square of the difference of the abscissas to the square of the difference of the ordinates and taking the square root of the sum.

10. The middle point of a segment.—Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in the plane and let $P(x, y)$ be the middle point of AB.

It is required to find the coördinates of the point P in terms of x_1, y_1, x_2, y_2.



Since $AP = PB$ we have $AM = MN$. But by §8, for all positions of A and B, $AM = x - x_1$ and $MN = x_2 - x$. That is, $x - x_1 = x_2 - x$. (1)

Also $NQ = QB$ and hence

$$y - y_1 = y_2 - y. \quad (2)$$

Solving (1) and (2) for x and y,

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2} \quad [2]$$

Hence we have:

To find the coördinates of the middle point of a segment find the half-sums of the coördinates of the end-points.

SIGHT WORK

1. Find the number values of the projections on the two axes of the following segments: (2, 3), (7, 9); (− 4, 6), (1, − 9); (0, 7), (3, − 9).

2. Find the number values of the projections on a line parallel to the x-axis of the segments: (− 4, − 2), (− 1, 7); (1, − 7), (3, 8); (− 1, − 2), (− 9, − 1).

3. Find the number values of the projections on a line parallel to the y-axis of the segments: (− 5, − 4), (− 2, − 9); (9, − 3), (2, − 1); (− 5, − 8), (− 2, − 7).

4. If $A(x_1, y_1)$, $B(x_2, y_2)$ are any two points in the plane, is it correct to write $\sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2}$ or $\sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2}$ as the formula giving the distance between the two points? Give reasons for your answer.

5. Does the formula for finding the distance between two points apply when both points are on the x-axis? when they are on the y-axis?

6. What is the distance between two points if they have the same ordinate? if they have the same abscissa?

7. Find the distances between the following pairs of points: (4, 0), (− 6, 0); (− 8, 0), (3, 0); (− 5, 0), (− 12, 0).

8. Find the distances between the following pairs of points: (0, 7), (0, 3); (0, − 4), (0, 5); (0, − 8), (0, − 12).

9. Find the distances between the following pairs of points: (0, 0), (6, 8); (0, 0), (− 6, 8); (0, 0), (− 6, − 8); (0, 0), (6, − 8).

10. Find the distances between the following pairs of points: (1, 2), (4, 6); (− 2, 3), (4, − 5); (− 1, − 2), (− 13, 3).

11. Find the middle point of each of the following segments: (5, 0), (7, 0); (− 3, 0), (13, 0); (− 1, 0), (− 15, 0).

12. Find the middle point of each of the following segments: (0, 3), (0, 17); (0, − 3), (0, 11); (0, − 3), (0, − 19).

13. Find the middle points of the following segments: (1, 2), (8, 5); (− 3, 7), (2, − 3); (3, − 9), (− 8, 1); (12, − 4), (− 6, 8); (a, 3), (4, b).

EXERCISES

1. Find the distance between the points (− 3, 1) and (6, 5). Draw a figure to show the position of these points and construct the right triangle used in finding the distance between them.

2. Find the distance between each of the following pairs of points: (3, 4), (7, 2); (9, − 1), (8, − 4); (− 3, 4), (2, − 9); (1, 0), (8, − 2); (− 8, − 7), (3, 2).

3. Show that $x^2 + y^2$ represents the square of the distance from the origin to the point (x, y).

4. If $x^2 + y^2 = 16$, what can be said of the point (x, y)? What is the locus of this point as x and y vary subject to the condition of this equation?

5. Find the lengths of the sides of a triangle whose vertices are (1, 5), (4, − 1), and (− 2, 0).

6. Show that the triangle whose vertices are (6, − 2), (− 2, 2), and (1, − 2) is isosceles.

7. Determine whether the triangle whose vertices are (1, 1), (6, 7), (− 4, 5) is right angled.

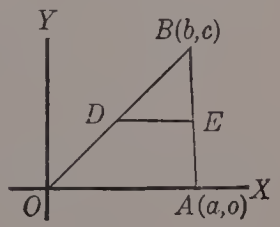
8. The diagonals of a square whose sides are $2a$ meet at the origin of the system of coördinates. Find the coördinates of the vertices of the square, (a) if the sides of the square are parallel to the axes, (b) if the diagonals of the square lie on the axes.

9. If the sum of the projections on the x-axis of the segments AB, BC, CD, DE is equal to zero, compare the abscissas of A and E.

10. If the sum of the projections of AB, BC, CD, DE, EF on each axis is zero, what can be said of the points A and F?

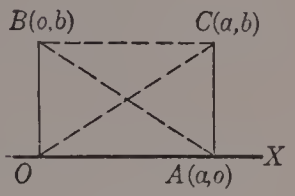
11. Show that the segment connecting the middle points of two sides of a triangle is equal to one-half the third side.

SUGGESTION: Place the triangle as shown in the figure. Then find, (a) the coördinates of D and E, (b) the length of DE.



12. Show that the diagonals of a rectangle are equal in length.

SUGGESTION: Place the rectangle as shown in the figure.



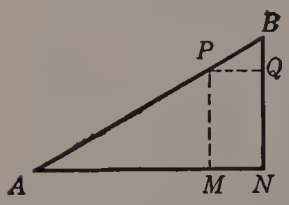
13. Show that the diagonals of a rectangle bisect each other.

14. Find the lengths of the sides of the triangle $A(1, 3)$, $B(4, 7)$, $C(- 1, - 3)$. Arrange the angles of this triangle in the order of their magnitude.

SUGGESTION: "If two sides of a triangle are unequal the angles opposite them are unequal, etc."

11. Point dividing a segment in a given ratio.—Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end-points of a given segment and let $P(x, y)$ be a point on AB such that

$$AP : PB = m : n,$$



where $m : n$ is a given ratio.

It is required to find the coördinates of the point P in terms of x_1, y_1, x_2, y_2, m, n.

Draw BN and PM parallel to the y-axis, and AN and PQ parallel to the x-axis. Then AMP and PQB are similar triangles, and $AM : PQ = AP : PB = m : n$.

But $AM = x - x_1$ and $PQ = x_2 - x$.

Hence
$$\frac{AM}{PQ} = \frac{x - x_1}{x_2 - x} = \frac{m}{n}. \tag{1}$$

In the same manner we obtain

$$\frac{y - y_1}{y_2 - y} = \frac{m}{n}. \tag{2}$$

Solving (1) and (2) for x and y respectively, we have

$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n} \tag{3}$$

If P is the middle point of the segment AB, then

$$m = n, \text{ and } x = \frac{x_2 + x_1}{2}, \; y = \frac{y_2 + y_1}{2}.$$

SIGHT WORK

1. Find the point two-thirds of the distance from $(-3, 1)$ to $(4, 10)$.

2. Find the middle points of the three sides of the triangle $A(-2, 2)$, $B(6, 0)$, $C(4, 4)$. Find the coördinates of the point two-thirds of the distance from each vertex to the middle point of the opposite side.

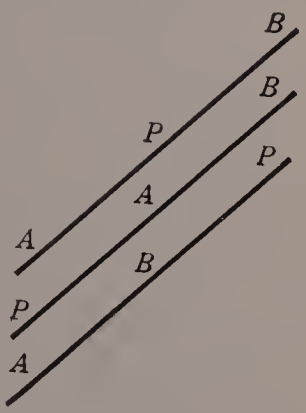
3. One end of a line segment is at the point $(1, 3)$, and the point one-fourth of the distance to the other end is $(3, 4)$. Find the coördinates of the other end of the segment.

12. External division of a segment.—If the ratio $AP : PB$ is negative then the point P does not lie on the segment AB, since AP and PB are opposite in quality. The student may readily satisfy himself of the truth of the following statements:

(a) If $AP : PB$ is positive, P lies between A and B.

(b) If $AP : PB$ is negative, and numerically less than one (that is, AP is shorter in length than PB), then the points are in the order PAB.

(c) If $AP : PB$ is negative, but greater numerically than one, then the points are in the order ABP.



SIGHT WORK

1. If the points A and B are $(-7, 5)$, $(5, -3)$, find the coördinates of P so that $AP : PB = -\frac{1}{4}$; also find the coördinates of P so that $AP : PB = -4$.

2. Find the coördinates of the point P such that $AP : PB = 1 : 14$ where A is the point $(-7, -5)$ and B is $(8, 10)$.

3. Using A and B of the preceding example, find the point P such that $AP : PB = 14 : 1$.

4. If the point P lies on the segment AB but approaches the point A as a limit, what limit does $AP : PB$ approach?

5. If the points are in the order PAB and P approaches A as a limit, what limit does $AP : PB$ approach?

6. If the points are in the order PAB and P recedes indefinitely far from A, what limit does $AP : PB$ approach?

7. If the points are in the order ABP and P recedes indefinitely far from B, what limit does $AP : PB$ approach?

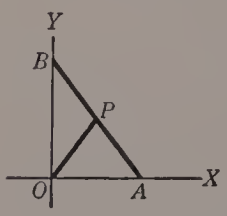
8. Describe the change in $AP : PB$ if the points are in the order APB and if P approaches B as a limit.

9. Describe the change in $AP : PB$ if the points are in the order ABP and if P approaches B as a limit.

EXERCISES

Group A

1. Show that the middle point of the hypotenuse of a right triangle is equidistant from the vertices.



SUGGESTION: Place the triangle as shown in the figure. Let $OA = a$ and $OB = b$. Then find (a) the middle point P of AB, (b) the length of OP, (c) the length of AB.

2. Find the length of the line segment connecting the points $(a,\ a + b\sqrt{3})$ and $(a + b,\ a)$.

3. Decide whether the triangle whose vertices are the points $A\ (1, 3)$, $B(-4, 2)$, $C(-8, 6)$ is a right triangle.

4. Arrange in order of magnitude the sides of the triangle $A(-9, 2)$, $B(4, 7)$, $C(4, -6)$. Also arrange the angles in their order of magnitude.

5. Find the length of the segment connecting the middle points of the sides AB and AC of the triangle $A(-1, -7)$, $B(7, -3)$, $C(8, 6)$.

6. Find the lengths of three medians of the triangle in Example 5.

7. If the segment $A(6, y_1)$, $B(x_2, -1)$ is bisected by the point $C(3, 9)$, find the values of y_1 and x_2.

8. Find the coördinates of C if it divides the segment $A(-1, -2)$, $B(3, -10)$ so that $AC : CB = 3:7$.

9. If the segment $A(x_1, 4)$, $B(2, y_2)$ is divided by $C(4, 6)$ so that $AC : CB = 2 : 5$, find the values of x_1 and y_2.

10. If $(x - a)^2 + (y - b)^2 = 36$, find the distance from the point (a, b) to the point (x, y). If (a, b) is a fixed point, what is the locus of (x, y)?

Discuss the problem of finding the equation of a circle with given center and given radius.

Group B

1. Find the length of the line segment connecting the points $(4\sqrt{3}, 4)$ and $(1, -\sqrt{3})$.

2. Find the length of the line segment connecting the points $(a\sqrt{b}, c)$ and $(c, -a\sqrt{b})$.

3. "If a median of a triangle is equal to one-half the side on which it is drawn, the triangle is right-angled." Use this proposition to decide whether the triangle $A(5, 1)$, $B(1, 4)$, $C(1, 1)$ is right-angled at C.

4. Arrange the angles of the triangle $A(-2, -3)$, $B(4, 7)$, $C(9, -5)$ in the order of their magnitude.

5. Find the length of the segment connecting the middle points of the sides AB and BC of the triangle $A(6, -3)$, $B(4, 8)$, $C(-4, 5)$. Also find the length of AC and compare.

6. Find the lengths of the medians of the triangle in Ex. 5.

7. If the segment $A(3, y_1)$, $B(x_2, 12)$ is bisected by $P(5, 7)$, find the values of y_1 and x_2.

8. Find the coördinates of P if it divides the segment $A(1, 9)$, $B(-7, 4)$ so that $AP : PB = 3 : 11$.

9. If the segment $A(4, -9)$, $B(-4, y_2)$ is divided by $P(x, 7)$ so that $AP : PB = 2 : 7$, find x and y_2.

10. If $(x-a)^2 + (y-b)^2 = 64$, what is the distance between the points (x, y) and (a, b)? If (a, b) is a fixed point, what is the locus of the point (x, y)?

Group C

1. Find the length of the segment joining the points $(3\sqrt{5}, 2)$ and $(4, -6\sqrt{5})$.

2. Find the length of the segment joining the points $(a\sqrt{b}, 2a)$ and $(4c, -2c\sqrt{b})$.

3. Find the length of the median on the side AB in the triangle $A(x_1, 0)$, $B(0, y_1)$, $C(0, 0)$ and verify that the median is equal to one-half the hypotenuse.

4. Decide whether the triangle $A(1, 7)$, $B(-7, 3)$, $C(2, -8)$ has an obtuse angle. Also arrange the angles of the triangle in the order of their magnitude.

5. Find the lengths of the medians of the triangle $A(3, 5)$, $B(-7, -3)$, $C(8, -4)$.

6. Find the length of the segment connecting the middle points of the sides AB and AC in the triangle of Ex. 5. Compare this with the length of the side BC.

7. The segment $A(-1, y_1)$, $B(x_2, -3)$ is bisected by the point $P(3, 4)$. Find the values of y_1 and x_2.

8. Find the coördinates of P if it divides the segment $A(1, -9)$, $B(8, -3)$ so that $AP : PB = 5 : 7$.

9. If the segment $A(x_1, 4)$, $B(2, 8)$ is divided by $P(2, y)$ so that $AP : PB = 2 : 5$, find x_1 and y.

10. If $(x-a)^2 + (y-b)^2 = 100$, what is the distance between (x, y) and (a, b)? If (a, b) is a fixed point, what is the locus of (x, y)?

MISCELLANEOUS PROBLEMS ON CHAPTER I

1. An equilateral triangle whose sides are a is so placed that one side lies on the x-axis with its left end-point at the origin and the third vertex is above the x-axis. What are the coördinates of the vertices?

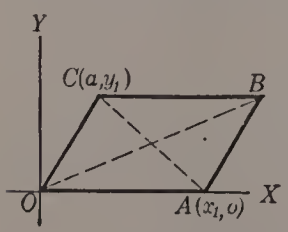
2. What are the coördinates of the vertices of the triangle in Ex. 1 if one side is on the x-axis, with its middle point at the origin, the third vertex being on the positive side of the x-axis?

3. Is either of the triangles (1, 2), (3, 4), (− 1, 4); (3, 0), (6, 4), (− 1, 3) a right triangle?

4. Find the perimeter of the quadrilateral $ABCD$ if its vertices are $A(1, 4)$, $B(7, 1\frac{1}{2})$, $C(-1, -4\frac{1}{2})$, $D(-11, -1)$.

5. Show that the points (8, 9), (10, 7), (− 4, − 7), (− 6, − 5) lie on a circle whose center is (2, 1).

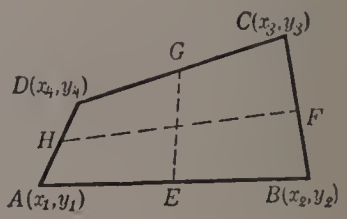
6. Find the distance between the points $(a - b, c - d)$, $(b - a, d - c)$. Also find the middle point of the segment joining these points.



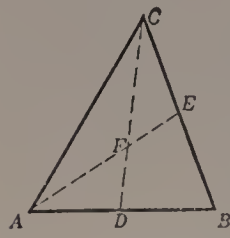
7. Prove that the diagonals of a parallelogram bisect each other.

SUGGESTION: Show that in the figure, if the coördinates of C are a, y_1 and those of A are x_1, 0, then the coördinates of B are $a + x_1$, y_1. Then find the middle points of the segments OB and AC.

8. Prove that the segments connecting the middle points of the opposite sides of a plane quadrilateral bisect each other.



SUGGESTION: Suppose the vertices of the quadrilateral are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$. Find the coördinates of E, F, G, H. Then find the middle points of EG and FH.



9. Prove that the three medians of a triangle meet in a point which is two-thirds the distance from a vertex to the middle point of the opposite side.

SUGGESTION: Suppose the vertices of the triangle are $A(-a, 0)$, $B(a, 0)$, and $C(b, c)$. Find the coördinates of D and E, the middle points of AB and BC. Then find F such that $AF:FE = 2 : 1$. Show that the coördinates of a point which divides CD in the ratio $2 : 1$ are the same as the coördinates of F.

CHAPTER II

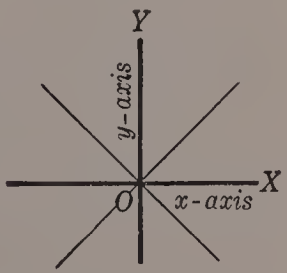
EQUATIONS AND THEIR LOCI

GRAPHS of simple equations are studied in beginning algebra and we will approach the more formal treatment of curves and their equations through a review and extension of this elementary work.

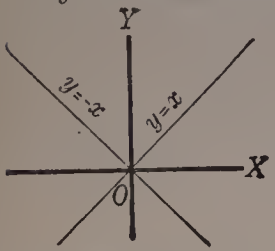
13. Geometric loci defined by means of equations.—The locus of a point is defined by giving some relation of the point to given points or lines. When a locus is defined by means of an algebraic equation the point is required to move so as to maintain a certain relation between its coördinates, the relation being stated in the equation.

Example 1. Find the locus of a point which is equidistant from the coördinate axes.

SOLUTION: The locus consists of the two lines shown in the figure. One line bisects the first and third quadrants, and the other bisects the second and fourth quadrants.



Example 2. Find the locus defined by the equation $y = x$ and also the locus defined by $y = -x$.



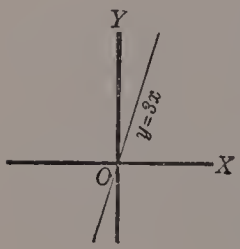
SOLUTION: The equation $y = x$ states that the ordinate and abscissa of every point on the locus must be equal. Not only must the point be equally "distant" from the axes, but it must be on the positive side of both axes or on the negative side of both axes. Hence the locus is the line marked $y = x$ in the figure.

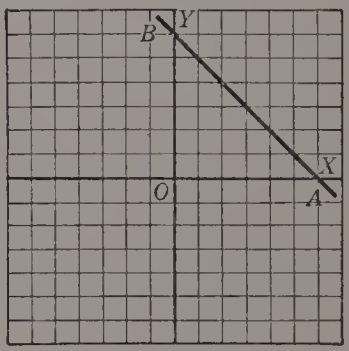
In the locus of $y = -x$, the point must be equally "distant" from the axes and further, if it is on the positive side of one axis, it must be on the negative side of the other.

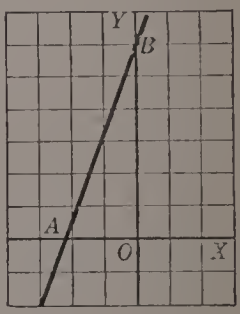
Example 3. Find the locus of the equation $y = 3x$.

SOLUTION: The given equation states that the ordinate of the point is 3 times the abscissa. That is, the distance from the x-axis is 3 times that from the y-axis. The line shown in the first figure is the locus.

Example 4. Find the locus of a point which moves so that the sum of its distances from the coördinate axes is 6.



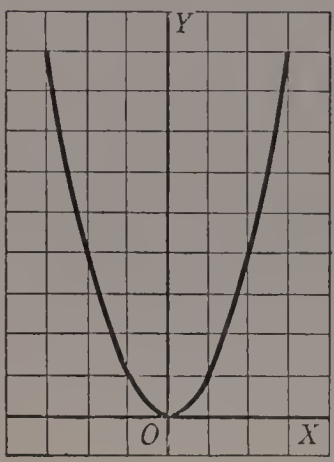




SOLUTION: If in this problem "distance" is made to include direction or sign as in a system of coördinates, then the condition of the locus may be stated by means of the equation $x + y = 6$. The locus of this equation is the complete line shown in the second figure.

Example 5. Find the locus of a point which moves so that twice its distance from the x-axis minus 5 times its distance from the y-axis equals 12.

SOLUTION: If "distance" includes direction, then the condition of the locus may be stated by means of the equation $2y - 5x = 12$, and the locus is the complete line AB shown in the third figure.



Example 6. Find the locus of the equation $y = x^2$.

SOLUTION: This equation is satisfied by the following pairs of values:

x	0	1	2	3	-1	-2	-3
y	0	1	4	9	1	4	9

Plotting the points determined by these pairs of values and drawing a smooth curve through them, we have the curve as shown.

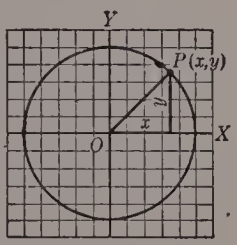
Example 7. Find the locus of $x^2 + y^2 = 25$.

SOLUTION: This equation is satisfied by the following pairs of values:

x	0	0	5	−5	3	3	−3	−3	4	4	−4	−4
y	5	−5	0	0	4	−4	4	−4	3	−3	3	−3

Plotting these points and drawing a smooth curve through them, we have a circle with center at the origin and radius 5.

That this equation represents a circle is at once evident from the fact that $x^2 + y^2$ is the square of the distance between (0, 0) and (x, y). If the square of this distance is 25, the point (x, y) is at a distance of 5 from (0, 0). That is, the equation represents the locus of a point whose distance from the origin is 5.



14. Relation between an equation and its locus.—The above examples illustrate the relation between an equation in two unknowns and the locus determined by it. Consider again the equation $x + y = 6$, or $y = 6 - x$. The letter x may be given any value in the system of real numbers, and for every such value there will be a definite value of y. It follows that this equation is satisfied by an infinite (unlimited) number of pairs of numbers. Each of these pairs of numbers determines a point, and it turns out that all these points lie on a straight line. This line and the equation have the following relations:

1. *Any two numbers which satisfy the equation are the coördinates of a point on the line.*
2. *The coördinates of any point on the line satisfy the equation.*

We may think of the point as moving along the line, the coördinates of the point changing so that their sum remains constant.

SIGHT WORK

1. Find the locus of a point which moves so that its distance from the x-axis is twice its distance from the y-axis. Find the equations of this locus, "distance" being used to include direction, or sign.

2. Find the locus of a point which moves so that its distance from the y-axis is twice its distance from the x-axis. What are the equations of this locus?

3. Write the equation of the locus of a point which moves so that its distance from the y-axis less its distance from the x-axis is 4. Note that in the equation "distance" from an axis includes its direction from the axis.

4. As in Example 6 on page 16, construct the locus of the equation $y^2 = x$. Compare this locus with the locus of $y = x^2$.

5. Find the locus of the equation $x^2 + y^2 = 64$.

6. State fully the relation between an equation and the coördinates of the points on its locus.

15. Variables and constants.—We have seen that in the equation $x + y = 6$ both x and y may take an unlimited number of different values, provided the sum of any pair of values is 6. For this reason the symbols x and y are called **variables.** The numeral 6 on the other hand represents only one value and is called a **constant.**

Consider now the equation $x + y = C$. In this equation C may take any value whatever. Nevertheless we think of this equation as the equation of a straight line. That is, we think of C as taking a fixed value, 6 for instance, while x and y are allowed to vary subject to the condition that their sum must remain equal to the fixed value of C. Hence for any fixed value of C the equation $x + y = C$ represents a straight line. Since C is supposed to assume a fixed value while x and y vary, C is regarded as a constant.

In the equation $Ax + By + C = 0$, the letters A, B, C are supposed to take fixed values while x and y vary subject to the condition that the equation must be satisfied. Hence A, B, C are constants and x and y are variables.

16. Functions and functional notation.—When the equation $x + y = 6$ is solved for y giving $y = 6 - x$, it is natural first to give values to x and then find the corresponding values of y. Similarly if the equation $2y - 5x = 12$ is solved for y, giving $y = \frac{12 + 5x}{2}$, it is natural first to give values to x and then find the corresponding values of y. In such cases x is called the **independent variable** and y the **dependent variable.**

Any expression containing a variable in such a way that the value of the expression is fixed when the value of the variable is fixed is called a **function** of that variable. Thus $6 - x$ and $\frac{12 + 5x}{2}$ are functions of x.

The symbol $f(x)$ is used to indicate any expression which is a function of x. This symbol is read "function of x" or simply "f of x."

17. Real and imaginary values of a function.—Functions such as $6 - x$ and $\frac{12 + 5x}{2}$ have real values no matter what real numbers are substituted for x. Other functions, such as $\sqrt{25 - x^2}$, have real values for some values of x and imaginary values for other values of x. Thus, if 4 is substituted for x in $\sqrt{25 - x^2}$, then $\sqrt{25 - x^2} = \sqrt{25 - 4^2} = \sqrt{9} = 3$, while if 6 is substituted for x, the value is $\sqrt{25 - 6^2} = \sqrt{-11}$, which is imaginary. It is clear that the value of $\sqrt{25 - x^2}$ is real for every value of x between -5 and 5 and also including these numbers, while it is imaginary for all other values of x. Since the coördinates of a point are real numbers it follows that imaginary values are not represented in our geometry.

SIGHT WORK

1. For what values of x does $\sqrt{x}$ have real values?
2. For what values of x does $\sqrt{x^2 + 4}$ have real values?
3. For what values of x does $\sqrt{x^3 + 8}$ have real values?

18. General graphic representation of a function.—Consider an equation of the form $y = f(x)$, in which y does not occur in the second member. It is possible that there may be no real value of x for which $f(x)$ has a real value. Thus in $y = \sqrt{-1-x^2}$ every real value of x gives an imaginary value of y. Such an equation has no real locus.

If there are real values of x for which the function has real values, then the equation represents a real geometric locus. That is, every pair of real values of x and y which satisfies the equation represents a point in the plane and all such points taken together constitute the locus.

Thus, in $y = \sqrt{25-x^2}$ all values of x between -5 and $+5$ and including these values give real values of y, while all other values of x give imaginary values of y. Hence the graph of this function lies entirely between these extreme values 5 and -5 of x.

In $y = \sqrt{x^2-25}$ the values of y are imaginary when x is between -5 and $+5$, while for all other real values of x they are real. Hence the graph of this equation has no point between these values of x.

The set of all values of x for which a function has definite real values constitute the **range** of the independent variable in that function, or the set of **permissible substitutions** in that function.

If a function has only one value for each permissible substitution, it is said to be **single valued.**

Many of the equations with which we have to deal are not of the simple form $y = f(x)$ where $f(x)$ is a single-valued function. Thus in the equation $x^2 + y^2 = 25$, y has two values for each permissible value of x except $x = -5$, $x = 5$. Such equations are indicated by the general symbol $f(x, y) = 0$. Thus $x^2 + y^2 = 25$ may be written $x^2 + y^2 - 25 = 0$.

In general for any equation $f(x, y) = 0$ we have:

If x_1, y_1 *satisfy the equation, then the point* (x_1, y_1) *is on the locus of the equation; and conversely, if* (x_1, y_1) *is a point on the locus, then* x_1, y_1 *satisfy the equation.*

19. Symmetry.—In the equation $y = x^2$, x occurs as a square only; that is, x^2 occurs but no other power of x. Hence two values of x, equal numerically but opposite in sign, give the same value of y.

Thus, this equation is satisfied by the following pairs of values:

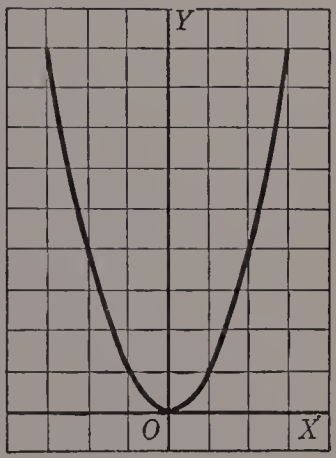
$$\begin{cases} x = 1 \\ y = 1 \end{cases} \quad \begin{cases} x = -1 \\ y = 1 \end{cases} \quad \begin{cases} x = 2 \\ y = 4 \end{cases} \quad \begin{cases} x = -2 \\ y = 4 \end{cases} \quad \begin{cases} x = 3 \\ y = 9 \end{cases} \quad \begin{cases} x = -3 \\ y = 9 \end{cases}$$

In general, if x_1, y_1 satisfy this equation, then $-x_1$, y_1 will also satisfy it.

This means that the locus is symmetrical with respect to the y-axis.

If y is involved as a square only, then for the same reason the locus is symmetrical with respect to the x-axis. Thus $y^2 = x$ is symmetrical with respect to the x-axis.

In general if both x and y are involved as even powers only, the locus is symmetrical with respect to both axes. In this case the locus is also symmetrical with respect to the origin. Thus the locus of $x^2 + y^2 = 25$ is a circle which is symmetrical with respect to both axes and also with respect to the origin.



EXERCISES

1. Construct the locus of $y = 2x^2$. Discuss this locus (a) as to the values x may take, (b) as to the values y may take, (c) as to symmetry.

2. Construct the locus of $y^2 = 4x$. Discuss this locus, (a) as to the values x may take, (b) as to the values y may take, (c) as to symmetry.

3. Construct the locus of $x^2 + y^2 = 49$. Discuss this locus, (a) as to the values x may take, (b) as to the values y may take, (c) as to symmetry.

4. Discuss the symmetry of the locus of $6x^2 + 9y^2 = 54$.

5. Discuss the locus of the equation $3x^2 - 7y^2 = 14$ as to symmetry.

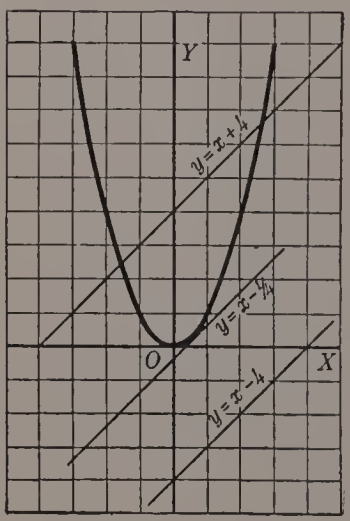
6. Discuss as to symmetry the locus of each of the following equations. (a) $y^2 = 4x^2 + 9x + 2$, (b) $y = 4x^2 + 9x + 2$, (c) $y^2 = 9x - 3$.

20. Intersection of loci.—If the loci of two equations meet in a point whose coördinates are x_1, y_1, then x_1, y_1 must satisfy both equations, and conversely if two equations are satisfied by x_1, y_1 then the point (x_1, y_1) must lie on both loci.

Hence we have the rule:

To find the intersection points of two loci, solve their equations simultaneously.

It is clear that if the two equations have one or more simultaneous real solutions, then the loci will meet; and if the solutions are all imaginary, the loci do not meet.



Example 1. Find the intersection points of the loci of $y = x + 4$ and $y = x^2$.

SOLUTION: Substituting $y = x + 4$ in $y = x^2$ we have

$$x + 4 = x^2 \quad \text{or} \quad x^2 - x - 4 = 0.$$

Hence $x = \dfrac{1 \pm \sqrt{1 + 16}}{2} = 2.56$ and -1.56. See § 240.

That is, $x = 2.56$ and $x = -1.56$
$y = 6.56$ and $y = 2.44$

are the intersection points.

Example 2. Find the intersection points of the loci of $y = x - 4$ and $y = x^2$.

SOLUTION: Substituting and solving, we have

$$x = \frac{1 \pm \sqrt{-15}}{2}.$$

Since these values of x are imaginary the loci have no intersection point.

Example 3. Find the intersection points of the loci of $y = x - \frac{1}{4}$ and $y = x^2$.

SOLUTION: Substituting and solving, we have

$$x = \frac{1 \pm \sqrt{1 - 1}}{2} = \frac{1}{2}.$$

Hence the line meets the curve in only one point. From the graph it is apparent that the line is tangent to the curve.

Example 4. For what values of c will the line $x + y = c$ meet the circle $x^2 + y^2 = 25$ in two points? in one point? in no point?

SUGGESTION: Solving simultaneously, we have

$$x = \frac{c \pm \sqrt{50 - c^2}}{2}.$$

For what values of c is $50 - c^2$ positive? for what values is it zero? for what values negative? For which of these cases is the line tangent to the circle?

EXERCISES

1. Find the point on the locus of the equation $x^2 + 2x + y - 24 = 0$ whose abscissa is 3.

2. Find the coördinates of the points at which the locus of $x^2 + 2x + y - 24 = 0$ crosses the x-axis.

SUGGESTION: At such points the value of y is zero.

3. Find the point of intersection of the locus of $3x - 2y = 5$ with the locus of $x + 4y = 6$, and also with the locus of $x + 2y = 7$.

4. Find the intersections of the loci of $x^2 = 3y$ and $3y = x + 6$.

5. Determine the four points of intersection of the loci of $x^2 + y^2 = 25$ and $9x^2 - 16y^2 = 0$.

6. What are the vertices of the triangle formed by the loci of $x - 4y + 3 = 0$, $y = 6x - 5$, and $5x + 3y = 31$?

7. Find the distance between the two intersections of the loci of $x^2 + y^2 = 100$ and $2y = x + 10$.

8. Find the intersection of the loci of $x^2 = 2y$ and $2x = y + 2$. Construct the graphs to show the relation of these loci.

9. Solve simultaneously, $y = x^2$ and $y = x - 6$. What kind of solution do you find? Discuss the intersection of the two loci. Construct a graph to support your discussion.

10. Find the distance between the points in which the locus of $4x + 3y = 12$ crosses the two axes.

SUGGESTION: What is the abscissa of the point in which any line cuts the y-axis?

11. Solve simultaneously the equations $y = 4x + b$ and $y = x^2 + 4x + 3$. What is the smallest value of b for which the loci of these equations will meet?

12. Solve simultaneously $y = mx$ and $y^2 = 4x - 4$. For what values of m will the loci of these equations meet?

13. Construct the graph of $y^2 = 4x - 4$, and on the same set of axes construct the graphs of $y = mx$ using the limiting values of m determined in Example 12.

14. Find the distance between the points of intersection of the loci of $x^2 + y^2 = 25$ and $3x - y = 0$.

15. Find the abscissas of the points in which the locus of $y = x^2 - 7x + 4$ meets the x-axis.

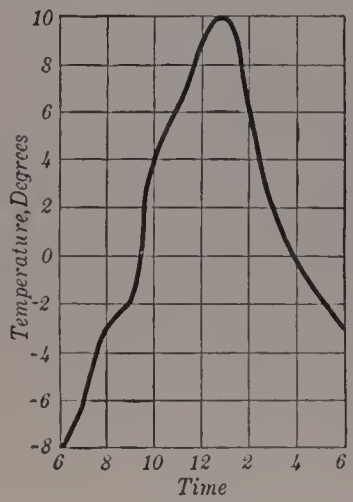
16. Find the ordinates of the points in which the locus of $y^2 + 7y - 9 = 3x$ meets the y-axis.

17. Find the coördinates of the points in which the locus of $x^2 + y^2 - 4x + 6y - 16 = 0$ meets the axes.

21. Graphs representing observational data.—Graphs are used extensively to represent data obtained from statistics or from direct observations. The following examples illustrate this use of the graph.

Example 1. The hourly observed temperatures for twelve hours were

	A.M.						P.M.						
Hour.....	6	7	8	9	10	11	12	1	2	3	4	5	6
Temperature...	− 8	− 6	− 3	− 2	4	6	9	10	6	2	0	− 2	− 3



These data are represented in the graph at the left.

In this graph the horizontal distances from the axis represent the number of hours after 6 A.M., and the vertical distances represent the temperatures. The points are plotted and then a smooth curve is drawn through them. This curve represents fairly closely the continuous change in temperature.

Example 2. The table on the next page is taken from a Bulletin of the U. S. Bureau of the Census. Construct a graph to represent the data given in each column.

Year	Estimated population of the U. S. Millions	Death rates in the registration area			
		All causes Per 1,000	Typhoid Per 100,000	Tubercu-losis Per 100,000	Malignant tumors Per 100,000
1900.....	76.0	17.6	35.9	201.9	63.0
1901.....	77.7	16.5	32.3	196.9	64.3
1902.....	79.4	15.9	34.3	184.5	65.1
1903.....	81.0	16.0	34.1	188.5	68.3
1904.....	82.6	16.5	31.7	200.7	70.2
1905.....	84.2	16.0	27.8	192.3	71.4
1906.....	85.8	15.7	31.3	180.2	69.1
1907.....	87.5	16.0	29.5	178.5	70.9
1908.....	89.1	14.8	24.3	167.6	71.5
1909.....	90.7	14.4	21.1	160.8	73.8
1910.....	92.3	15.0	23.5	160.3	76.2
1911.....	93.7	14.2	21.0	150.1	74.4
1912.....	95.1	13.9	16.5	149.7	77.1
1913.....	96.5	14.1	17.9	147.9	79.0
1914.....	97.9	13.6	15.5	147.2	79.6
1915.....	99.3	13.6	12.4	146.4	81.4
1916.....	100.8	14.0	13.3	142.1	82.1
1917.....	102.2	14.3	13.5	147.0	82.0
1918.....	103.6	18.1	12.6	150.0	80.3
1919.....	105.0	12.9	9.2	125.7	80.5
1920.....	106.4	13.1	7.8	114.2	83.4
1921.....	107.8	11.6	9.0	99.4	86.0

MISCELLANEOUS PROBLEMS ON CHAPTER II

1. A point moves so that its ordinate is always three greater than four times its abscissa. Write the equation of its locus.

2. Construct the graph of the equation found in Ex. 1.

3. From the equation $xy = 1$ find the values of y corresponding to the given values of x.

x	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	5	10
y										

Locate the points determined by these values of x and y and draw a smooth curve through them.

4. From the equation $xy = 1$, find the values of y corresponding to the negatives of the values of x given in Ex. 3, and then construct a curve through the resulting points.

5. Solve $y^2 = 4x$ simultaneously with each of the equations $x + y - 9 = 0$, $x + y + 1 = 0$, $x + y + 5 = 0$.

6. Construct on one set of axes the graphs of all the equations in Ex. 5 and interpret the solutions.

7. What limitations are there on the values of x and y in the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$? Within what rectangle must the locus of this equation lie?

8. What limitation is there on the values which x may take in the equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$? Construct the graph of this equation.

9. Find the lengths of the sides of the triangle formed by the lines $y = 4x + 10$, $3x = 7y + 5$, and $x + y = 5$.

10. Find the x-coördinates of the points of intersection of the loci of $x^2 + y^2 = 64$ and $y = \frac{3}{4}x + b$. For what values of b are these two values of x equal?

11. Write the equation of the locus of a point which moves so that the sum of the squares of its ordinate and abscissa is always equal to 4. On what curve does such a point move?

12. Solve $y = x^2$ simultaneously with each of the equations $y = x + 6$, $y = x - \frac{1}{4}$, $y = x - 6$. Construct the figures and interpret the solutions.

13. Solve $x^2 + y^2 = 25$ simultaneously with each of the equations $y = \frac{4}{3}x + 3$, $y = \frac{4}{3}x + 8\frac{1}{3}$, $y = \frac{4}{3}x + 10$. Construct the figures and interpret the solutions.

14. Find the intersection points of the loci of $x^2 + y^2 = 16$ and $9x^2 + 25y^2 = 225$.

15. Attempt to find intersection points of $x^2 + y^2 = 36$ and $9x^2 + 25y^2 = 225$. Interpret the result.

CHAPTER III

THE STRAIGHT LINE

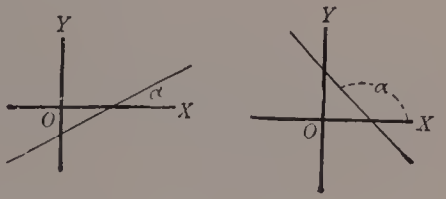
THE main purpose of this chapter is to prove that an equation of the first degree in two variables is the equation of a straight line, and to study properties of straight lines by means of such equations. The general form of the first degree equation is $Ax + By + C = 0$, in which A, B, C are constants. The location of the line in the coördinate plane is determined by the values of these constants.

22. Lines parallel to the axes.—If a line is perpendicular to the x-axis and meets it at a distance a from the origin, then $x = a$ for every point on the line.

Hence $x = a$ is the equation of a line parallel to the y-axis, the line being to the right or left of the axis according as a is positive or negative.

In the same way it follows that $y = b$ is the equation of a line parallel to the x-axis and at a distance b from it.

23. The slope of a line.—When we speak of the angle which a line makes with the x-axis, we mean the angle formed by the upper part of the line and the positive part of the axis. In the first figure the line makes an acute angle with the axis and in the second figure the angle is obtuse. In each case the angle is denoted by α.

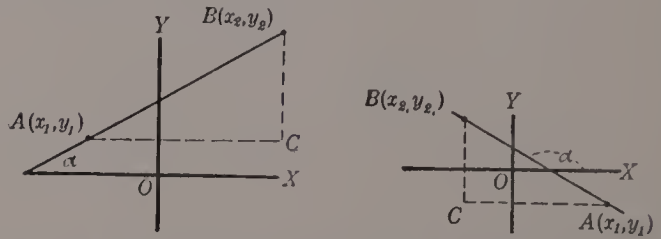


The **slope** of a line, usually denoted by m, is the tangent of the angle which the line makes with the x-axis; that is, $m = \tan \alpha$.

24. Parallel lines.—Since parallel lines make equal angles with the x-axis they have the same slope. Conversely, if two lines have the same slope, they make equal angles with the x-axis and are therefore parallel. Hence we have the proposition:

If two lines are parallel, they have the same slope; and conversely if two lines have the same slope, they are parallel.

25. Slope of a line through two points.—If a line passes through two given points, its slope may be expressed in terms of the coördinates of these points.



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two given points. It is easily verified that in either figure the slope of the line AB is,

$$\frac{CB}{AC} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence we have that the slope of the line through the points (x_1, y_1), (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad [4]$$

SIGHT WORK

1. Write the equations of the lines parallel to the y-axis which meet the x-axis in the points whose abscissas are $4, 7, -4, -7, a, -a$.

2. Write the equations of the lines parallel to the x-axis which meet the y-axis in the points whose ordinates are $5, 9, 1, -7, -3, -6, b, -b$.

3. Find the slope of the line through the origin and each of the following points: (a) (1, 5); (b) (− 3, 4); (c) (− 1, − 3); (d) (4, − 3).

4. What is the tangent of the angle which the line through the points (1, 3) and (5, 3) makes with the x-axis?

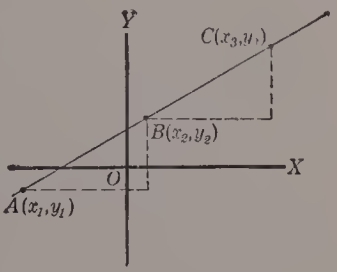
5. What angle does the line through the points (3, 1) and (3, 5) make with the x-axis?

26. Collinear points.—Since two lines having the same slope are parallel, or else coincident, it follows that if two lines have a point in common and have the same slope they coincide.

This gives a convenient test for collinearity of points. Thus, the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are collinear provided the lines AB and BC have the same slope, since they have the point B in common.

Hence, these points are collinear if

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \qquad [5]$$



The order in which the points are taken is immaterial. That is, these points are collinear provided the lines AC and BC have the same slope, or provided the lines AB and AC have the same slope.

Hence the test may be written,

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}, \text{ or } \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}, \text{ or } \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_3}{x_2 - x_3}.$$

SIGHT WORK

1. Find the slope of the line AB and of the line AC, where A, B, and C are the points (1, 1), (3, 4) and (5, 7), respectively. Are A, B, and C collinear?

2. Test for collinearity the following sets of points:

(a) (− 3, 4), (2, 3), (7, 2); (b) (− 4, − 6), (2, 0), (3, 1);
(c) (− 4, − 1), (5, 8), (1, 5); (d) (0, 5), (2, 1), (6, 7)

3. Which of the following pairs of points are on parallel lines?

(a) (4, 7), (2, 6) and (− 1, 9), (3, 7); (b) (− 2, 5), (6, 3) and (4, − 2), (2, 3); (c) (4, − 1), (2, 6) and (9, 11), (− 1, − 3); (d) (5, − 2), (− 2, 5) and (− 4, 9), (7, − 2); (e) (3½, 7), (− 2¼, 3½) and (6, − 12), (− 17, − 26); (f) (− 5, 3), (9, − 10) and (7, − 3½), (− 2, − 6½).

4. Find the slopes of the medians of the triangle whose vertices are: $A(2, 7)$, $B(-6, 5)$, $C(8, -3)$.

5. Find x_1 if the points $(x_1, 5)$, (7, − 3), and (− 2, 6) are collinear.

6. Find y_1 if the points (− 4, 7), (9, 2), and (6, y_1) are collinear.

EXERCISES

1. Find the slope of the line joining (a) $(-1, 2)$ and $(4, 5)$; (b) $(-3, 5)$ and $(2, -1)$; (c) $(5, -1)$ and $(-6, -6)$; (d) $(-3, 0)$ and $(0, 2)$.

2. What is the angle between the x-axis and the line passing through the points (a) $(1, 1)$ and $(4, 4)$; (b) $(-2, 0)$ and $(4, 6\sqrt{3})$; (c) $(1, 7)$ and $(5, 3)$?

3. Find the slope of the median from the vertex C of the triangle ABC where A, B, and C are the points $(-3, -4)$, $(5, 1)$, and $(-2, 6)$ respectively.

4. Find y so that the line passing through $(5, y)$ and $(9, 7)$ shall be parallel to the line passing through $(3, 1)$ and $(9, 5)$.

5. Find the slope of each side of the triangle $A(4, 2)$, $B(1, 7)$, $C(-3, 3)$.

6. Find the slope of the line joining the mid-points of AB and BC in the triangle $A(5, 1)$, $B(-3, 6)$, $C(0, -4)$; then find the slope of AC and compare these slopes.

7. A line is drawn through the points $(5, 1)$ and $(1, -1)$, and a second line is drawn through the points $(-3, 4)$ and $(5, 8)$; determine whether these lines are parallel.

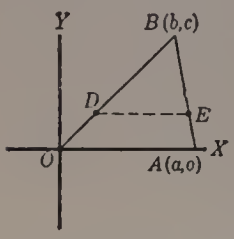
8. The line $A(7, 2)$, $B(0, y_1)$ has the same slope as the line $A(7, 2)$, $C(3, 6)$; find y_1.

9. The point $(8, y_1)$ is collinear with the two points $(1, 5)$ and $(3, 4)$; find y_1.

10. Find the point on the x-axis which is collinear with the points $(5, 6)$ and $(-4, -1)$.

11. Find the slopes of the medians of the triangle $A(4, -2)$, $B(-7, 5)$, $C(-3, -5)$.

12. The points P and Q divide the sides AB and AC of the triangle ABC so that $AP : PB = 3 : 10$ and $AQ : QC = 3 : 10$; find the slope of PQ and compare with the slope of BC if it is given that the vertices of the triangle are $A(-8, 1)$, $B(7, -2)$, $C(5, 12)$.



13. The vertices of a triangle are $O(0, 0)$, $A(a, 0)$ and $B(b, c)$; prove analytically that the line joining the middle points of OB and BA is parallel to OA and half as long as OA.

14. Prove that a line which divides two sides of a triangle in the same ratio is parallel to the third side.

SUGGESTION: Place the triangle as shown in the figure and find the coördinates of D and E, assuming that the ratio $OD : DB = m : n$.

27. The equation of a straight line.—Consider the first degree equation.

$$Ax + By + C = 0. \tag{1}$$

Let x_1, y_1 and x_2, y_2 be any two pairs of numbers which satisfy this equation.

Then
$$Ax_1 + By_1 + C = 0$$
$$Ax_2 + By_2 + C = 0.$$

Subtracting, $A(x_1 - x_2) + B(y_1 - y_2) = 0.$

Dividing and transposing,

$$\frac{y_1 - y_2}{x_1 - x_2} = -\frac{A}{B}.$$

Hence it follows from §25, that the slope of the line connecting the points $Q(x_1, y_1)$ and $R(x_2, y_2)$ is $-\frac{A}{B}$.

If x_3, y_3 satisfy (1) then by the same argument the slope of the line connecting $R(x_2, y_2)$ and $S(x_3, y_3)$ is also $-\frac{A}{B}$. Hence by §26, the points Q, R, S are collinear.

By the converse of this argument, if x_1, y_1 and x_2, y_2 satisfy equation (1) and if $T(x_4, y_4)$ lies on the line determined by $Q(x_1, y_1)$, $R(x_2, y_2)$, then equation (1) is satisfied by x_4, y_4.

It follows that for an equation $Ax + By + C = 0$ there is a straight line QR such that:

(*a*) *Every point whose coördinates satisfy the equation lies on the line.*

(*b*) *The coördinates of every point on the line satisfy the equation.*

Hence we say that the line QR is the locus of the equation $Ax + By + C = 0$, and that the equation is the equation of the line. For brevity we speak of the line $Ax + By + C = 0$.

Since $Ax + By + C = 0$ is a general equation of the first degree in x and y it follows that:

Every first degree equation in two variables is the equation of a straight line.

For this reason we speak of the equation as a **linear equation**.

28. Line through two points.—Let (x, y) be any point on the line through (x_1, y_1) and (x_2, y_2). Then by §26,

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \quad \text{or} \quad y - y_1 = (x - x_1)\frac{y_1 - y_2}{x_1 - x_2} \qquad [6]$$

Either of these forms is the equation of the straight line through the given points (x_1, y_1) and (x_2, y_2).

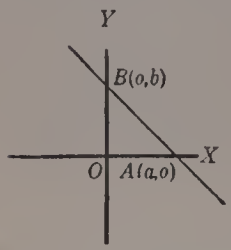
29. Line passing through a given point and having a given slope.—To find the equation of a line with a slope m passing through the point (x_1, y_1), let (x, y) be any point on the line. Then by §25,

$$\frac{y - y_1}{x - x_1} = m, \quad \text{or} \quad y - y_1 = m(x - x_1) \qquad [7]$$

30. Slope-intercept form of the equation of a straight line.—The distances (and directions) from the origin at which a line cuts the two axes are called the **intercepts** of the line on the axes. The intercepts may be distinguished as the **x-intercept** and the **y-intercept.** To find the equation of a straight line with slope m and y-intercept b, let (x, y) be any point on the line. Since the line cuts the y-axis a distance b from the origin it passes through the point $(0, b)$. Hence by §29 its equation is

$$\frac{y - b}{x - 0} = m, \quad \text{which reduces to } y = mx + b \qquad [8]$$

31. Equation of straight line in terms of its intercepts on the axes.—Let a and b be the x-intercept and the y-intercept respectively of a line. Then the line passes through the points $(a, 0)$, $(0, b)$. By §28 the equation of the line is



$$\frac{y - 0}{x - a} = \frac{0 - b}{a - 0} \text{ or } ay = ab - bx.$$

Transposing $-bx$ and dividing by ab,

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad [9]$$

This is called the **intercept** form of the equation.

EXERCISES

1. Find the equations of the straight line

(*a*) Passing through the origin, with slope 3.
(*b*) Passing through (3, − 1) and (2, 7).
(*c*) Having the intercepts − 5 and 4.
(*d*) With y-intereept − 2 and slope 3.
(*e*) Passing through (2, 2) and parallel to the y-axis.
(*f*) Having the y-intercept 4 and parallel to $2x - 3y = 1$.
(*g*) With slope 1 and y-intercept the same as that of the line $x - 2y = 6$.
(*h*) Through the point (5, 3) and interseeting $y = 2x + 2$ on the y-axis.

2. Determine the values of a, b, and m, as indicated, from the following equations.

(*a*) $y = 4x - 3$; (m, b).
(*b*) $\frac{x}{5} + \frac{y}{2} = 3$; (a, b).
(*c*) $2y = 3$; (m, b).
(*d*) $\frac{3}{5}x + \frac{2}{7}y = 1$; (a, b).
(*e*) $3x + 2y = 0$; (m, b).
(*f*) $x = 4$; (a).

3. Determine whether the three points (− 1, 2), (0, 3½), (4, 8½), lie upon the same straight line.

4. Find the equations of the sides of the triangle whose vertices are the points (1, − 1), (7, 1), (4, 5).

5. Find the middle points of the sides of the triangle of Ex. 4.

6. Find the equations of the medians of the triangle of Ex. 4.

7. Find the points of intersection of these medians in pairs.

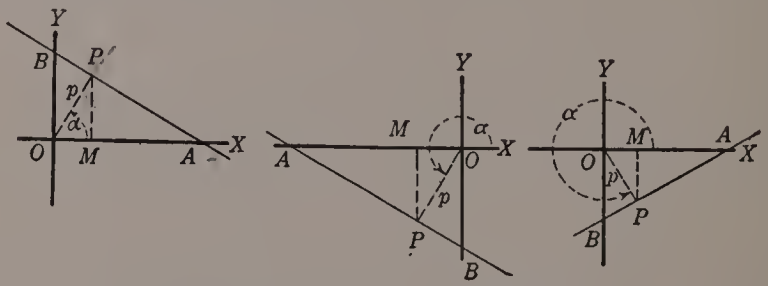
8. Find the equation of the line connecting the middle points of two sides of the triangle of Ex. 4, and prove that it is parallel to the third side of the triangle.

9. What is the equation of the line passing through the origin and through the intersection of the lines $x + 3y = 8$ and $2x - y = 9$?

10. The equations of the sides of a triangle are $y = x + 1$, $x + 2y = 8$, and $x + 8y = 8$. Find the equations of the lines passing through the vertices and parallel to the opposite sides.

11. Find the equation of the line through the intersection of $2x - 3y = 1$ and $2y - x = 1$, which makes an angle of 30° with the positive end of the x-axis.

32. Normal form of the equation of a straight line.—Let AB be any straight line, and let OP be the perpendicular from the origin upon it. Denote by α the angle which this perpendicular makes with the x-axis.



If $OP = p$, $OA = a$, and $OB = b$, then, in each figure,

$$\frac{p}{a} = \frac{OM}{p} = \cos\alpha \quad \text{and} \quad \frac{p}{b} = \frac{MP}{p} = \sin\alpha,$$

or

$$a = \frac{p}{\cos\alpha}, \quad b = \frac{p}{\sin\alpha}.$$

Substituting in $\frac{x}{a} + \frac{y}{b} = 1$ (§31) and reducing,

$$x\cos\alpha + y\sin\alpha = p \qquad [10]$$

It is left for the student to complete this proof for the case when $\alpha = 0$ or $\alpha = \pi$, when $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, when $p = 0$.

This is called the **normal form** of the equation of the straight line and $OP = p$ is said to be the **normal** upon the line.

Note that in this equation p is always positive and that α goes from 0° to 360°. If two lines are parallel, the angle α is the same for both provided they are on the same side of the origin; if the lines are on opposite sides of the origin, the α's differ by π as is seen in the first two figures.

33. Reducing an equation to the normal form.—The possibility of reducing any first degree equation in x and y to the normal form $x\cos\alpha + y\sin\alpha = p$ (see §32) depends upon the following proposition:

If two real numbers r and s are such that $r^2 + s^2 = 1$, then there is an angle α such that $\sin\alpha = r$ and $\cos\alpha = s$.

The proof depends upon the identity $\sin^2\alpha + \cos^2\alpha = 1$ and is left to the student.

Now consider $Ax + By + C = 0.$ (1)

Transposing and dividing by $\pm\sqrt{A^2 + B^2}$ we have

$$\frac{Ax}{\pm\sqrt{A^2+B^2}} + \frac{By}{\pm\sqrt{A^2+B^2}} = \frac{-C}{\pm\sqrt{A^2+B^2}} = 0. \quad (2)$$

But

$$\left(\frac{A}{\pm\sqrt{A^2+B^2}}\right)^2 + \left(\frac{B}{\pm\sqrt{A^2+B^2}}\right)^2 = \frac{A^2}{A^2+B^2} + \frac{B^2}{A^2+B^2} = \frac{A^2+B^2}{A^2+B^2} = 1.$$

Hence there is an angle α such that

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2+B^2}} \quad \text{and} \quad \sin\alpha = \frac{B}{\pm\sqrt{A^2+B^2}}.$$

If we let $\dfrac{-C}{\pm\sqrt{A^2+B^2}} = p$, then equation (2) may be written

$$x\cos\alpha + y\sin\alpha = p.$$

Use that sign of $\sqrt{A^2+B^2}$ which makes $p = -\dfrac{C}{\pm\sqrt{A^2+B^2}}$ positive.

That is, divide by $\sqrt{A^2+B^2}$ when C is negative and by $-\sqrt{A^2+B^2}$ when C is positive.

Example 1. Reduce $3x + 4y = 3$ to the normal form.

SOLUTION: $\sqrt{A^2+B^2} = \sqrt{3^2+4^2} = \sqrt{25} = 5.$

Hence the required equation is

$$\frac{3x}{5} + \frac{4y}{5} = \frac{3}{5}.$$

Example 2. Reduce $2x - 5y = -3$ to the normal form.

SOLUTION: $\sqrt{A^2+B^2} = \sqrt{2^2+(-5)^2} = \sqrt{(4+25)} = \sqrt{29}.$

Dividing by $-\sqrt{29}$, the required equation is

$$-\frac{2x}{\sqrt{29}} + \frac{5y}{\sqrt{29}} = \frac{3}{\sqrt{29}}; \text{ and } p = \frac{3}{\sqrt{29}}, \ \alpha = \cos^{-1}\left(\frac{-2}{\sqrt{29}}\right) = \sin^{-1}\left(\frac{5}{\sqrt{29}}\right).$$

EXERCISE

Reduce each of the equations $x + y = 4$, $x + y = -4$, $x - y = 4$, $x - y = -4$ to the normal form. Construct the figures and find the angle which each normal makes with the x-axis.

34. Summary of the forms of the equation of the straight line.—A straight line is determined by any one of the following sets of conditions:

(*a*) *two points on the line;*

(*b*) *a point on the line and its direction;*

(*c*) *the length of the perpendicular from the origin to the line and the direction of this perpendicular.*

If the two given points on the line are (x_1, y_1) and (x_2, y_2), then by §28 the equation of the line is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}. \qquad \text{(two-point form)} \tag{1}$$

If the two given points lie on the x-axis and y-axis respectively, that is, if their coördinates are $(a, 0)$, $(0, b)$, then by §31 equation (1) may be reduced to

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad \text{(intercept form)} \tag{2}$$

If the line passes through the point (x_1, y_1) and has a slope m, then by §29 its equation is

$$y - y_1 = m(x - x_1). \qquad \text{(point-slope form)} \tag{3}$$

If the given line passes through a given point $(0, b)$ (on the y-axis) and has a slope m, then by §30 its equation is

$$y = mx + b. \qquad \text{(slope-intercept form)} \tag{4}$$

If the length of the perpendicular from the origin to the line is p and the angle which this perpendicular makes with the x-axis is α, then by §32 its equation is

$$x \cos \alpha + y \sin \alpha = p. \qquad \text{(normal form)} \tag{5}$$

Attention to the forms of the five equations of the straight line given above and to the conditions which give rise to each form will greatly facilitate the solution of problems on the straight line.

35. Conditions which determine lines given indirectly.— Frequently one or both of the conditions which determine a line are given indirectly. Thus it may be required to find the equation of a line through a given point and parallel to a given line. In this case the slope of the given line must be found first and then equation (3) of page 36 is used. Again it may be required to find the equation of a line through the intersection point of two given lines and perpendicular to a third line. In this case the slope may be found by using §38 and the intersection point may be found as in §20. Then equation (3) of page 36 may be used. For another method see §44.

EXERCISES

1. Write the equation $4x - 3y = 10$, (a) in the slope-intercept form; (b) in the intercept form; (c) in the normal form; (d) in the two-point form. For this last form use the points whose abscissas are 1 and 4.

2. Write the equations of the lines which are determined by the following conditions.

(*a*) x-intercept 5 and slope 2.

(*b*) y-intercept -4, the line passing through the point (5, 2).

(*c*) Through the point (3, 4), the perpendicular from the origin to the line making an angle of 30° with the x-axis.

(*d*) Through the point (3, 7) if the intercepts on the axes are numerically equal, the y-intercept being positive and the x-intercept negative.

3. Find the equation of the line through the point (6, 4) which has equal positive intercepts on the axes.

4. What are the equations of the diagonals of a rectangle whose vertices are (0, 0), (a, 0), (0, b) and (a, b)?

5. A line through (3, 7) is such that the portion of it between the axes is bisected by that point. Find the equation of the line.

6. Find the area of the triangle formed by the axes and the line $3y + 4x = 48$.

7. Write the equation of the line whose intercepts on the axes are -3, 7; find the distance from the origin to this line.

8. Find the equation of the line through $(-4, -7)$ which has equal intercepts on the axes.

36. Parameters of a straight line.—Consider the following three forms of the equation of a straight line.

$$(1)\ y = mx + b, \quad (2)\ \frac{x}{a} + \frac{y}{b} = 1, \quad (3)\ x\cos\alpha + y\sin\alpha = p.$$

The five quantities a, b, m, p, α are called **parameters** of the straight line. The parameters a and b are the intercepts on the axes, m is the slope, p the length of the perpendicular from the origin upon the line, and α the angle which this perpendicular makes with the x-axis. To determine these parameters for a given equation reduce to the appropriate form.

Thus, to find a and b reduce to form (2); to find m and b reduce to form (1); and to find p and α reduce to form (3).

Example. Find the five parameters of the line $3x - 7y = 9$.

SOLUTION: Dividing by 9 we have $\frac{1}{3}x - \frac{7}{9}y = 1$

or $$\frac{x}{3} + \frac{y}{9/7} = 1.$$ Hence, $a = 3$ and $b = -\frac{9}{7}$.

To find m reduce the equation to form (1) by solving for y.

Then, $$y = \frac{3x}{7} - \frac{9}{7},$$ and hence, $m = \frac{3}{7}$.

To find α and p reduce to form (3):

Then, $$\frac{3}{\sqrt{58}}x - \frac{7}{\sqrt{58}}y = \frac{9}{\sqrt{58}}.$$

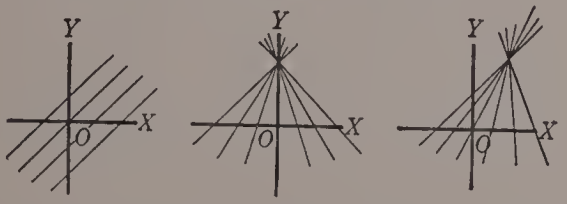
Hence, $$p = \frac{9}{\sqrt{58}} \quad \text{and} \quad \alpha = \cos^{-1}\left(\frac{3}{\sqrt{58}}\right) = \sin^{-1}\left(\frac{-7}{\sqrt{58}}\right).$$

EXERCISES

Find the five parameters of each of the following lines.

1. $3x - 2y = 4$
2. $x + 9y - 7 = 0$
3. $4x - 5y = 10$
4. $2x - 9y - 12 = 0$
5. $5x + 8y + 16 = 0$
6. $-7x + 4y = 18$
7. Find the parameters of the line $x\cos 45° + y\sin 45° = 10$.
8. Find the parameters of the line through the points $(-4, -2)$, $(3, -7)$.
9. Find the parameters of a line through the point $(3, 7)$ with slope $5/9$.

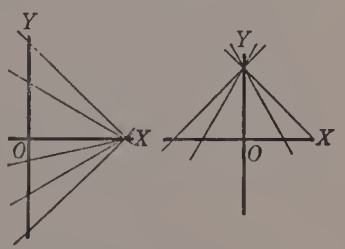
37. Families of lines.—If in the equation $y = mx + b$, m remains fixed while b takes different values, then the corresponding lines will be parallel. The set of all lines represented by $y = mx + b$ when m is fixed while b takes all real values is called **a family of lines** whose parameter is b.



If b remains fixed while m takes all real values, then the corresponding lines all pass through the fixed point $(0, b)$. This gives another family of lines whose parameter is m. Each of these families is called a **single parameter family,** because all the lines of the family are obtained by varying one parameter.

If in $y - y_1 = m(x - x_1)$, m takes all values while the point (x_1, y_1) remains fixed, we have a single parameter family of lines all passing through (x_1, y_1).

If in $\frac{x}{a} + \frac{y}{b} = 1$, a is fixed while b takes all real values, then we have a family of lines through the point $(a, 0)$, the value of b determining the y-intercept for each line. Similarly if b remains fixed while a takes different values, then we have a family of lines through the point $(0, b)$.

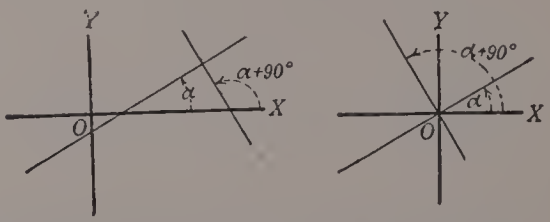


The form $y = mx + b$ can be used to represent any line except one which is parallel to the y-axis. The form $\frac{x}{a} + \frac{y}{b} = 1$ can be used to represent any line except one which is parallel to either of the axes.

SIGHT WORK

Discuss the family of lines represented by $x \cos \alpha + y \sin \alpha = p$, (a) when α is fixed and p varies, (b) when p is fixed and α varies.

38. Perpendicular lines.—We have already seen (§ 24) that lines which have the same slope are parallel. If two lines are perpendicular to each other, their slopes are subject to a simple relation. For simplicity construct lines through the origin parallel to the given lines as in the second figure.



From trigonometry [see § 252 (9)] we know that

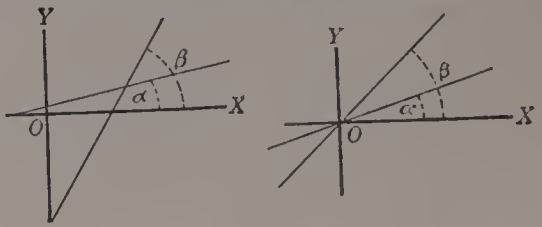
$$\tan(\alpha+90°) = -\frac{1}{\tan\alpha}$$

Hence $\tan(\alpha+90°)\tan\alpha = -1$ and we have the proposition:

If m_1 *and* m_2 *are the slopes of two lines at right angles to each other,*

then $$m_1 = -\frac{1}{m_2}, \quad \text{or} \quad m_1 m_2 = -1 \qquad [11]$$

39. Angle between two lines.—The tangent of the angle between two lines may be expressed in terms of the slopes of the two lines. For simplicity construct lines through the origin parallel to the given lines as in the second figure.



Then in the figures the angle between the given lines is $\beta - \alpha$,

and [§ 252 (8)] $$\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha}.$$

If $\tan \alpha = m_1$ and $\tan \beta = m_2$

$$\tan(\beta - \alpha) = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad [12]$$

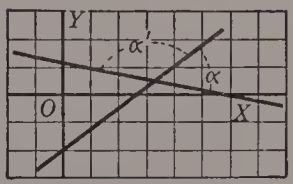
If $\tan(\beta - \alpha)$ as determined by [12] is positive, the angle $(\beta - \alpha)$ is acute and if $\tan(\beta - \alpha)$ is negative, the angle is obtuse. From §252 (8), it follows that $\frac{m_2 - m_1}{1 + m_1 m_2}$ and $\frac{m_1 - m_2}{1 + m_1 m_2}$ are the tangents of the two angles formed with each other by the lines whose slopes are m_1 and m_2. This is best understood from an example.

Example 1. Find the angles formed by the lines

$$3x - 4y = 8 \quad \text{and} \quad x + 5y = 6.$$

SOLUTION: By the method of §36 the slopes are found to be $\frac{3}{4}$ and $-\frac{1}{5}$.

The formula then gives as the tangent of the angle formed by the lines



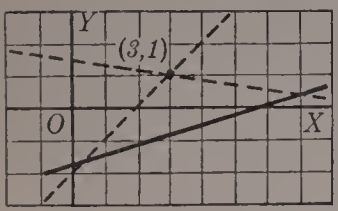
$$\frac{\frac{3}{4} - \left(-\frac{1}{5}\right)}{1 + \frac{3}{4}\left(-\frac{1}{5}\right)} = 1\frac{2}{17} \text{ or } \frac{\left(-\frac{1}{5}\right) - \frac{3}{4}}{1 + \left(-\frac{1}{5}\right)\frac{3}{4}} = -1\frac{2}{17}$$

depending on the order in which the slopes are substituted in the formula. The positive result gives the tangent of the acute angle α shown in the figure and the negative result gives the tangent of the obtuse angle α'.

Example 2. A line through the point (3, 1) makes with the line $2x - 6y = 11$ an angle whose tangent is $\frac{1}{2}$. Find the equation of the line.

SOLUTION: The slope of the given line is $\frac{1}{3}$. Let m be the slope of the required line.

Then, $\frac{\frac{1}{3} - m}{1 + \frac{1}{3}m} = \frac{1}{2}$, and also $\frac{m - \frac{1}{3}}{1 + \frac{1}{3}m} = \frac{1}{2}$.



Solving these equations for m we have

$$m = -\frac{1}{7} \text{ and } m = 1.$$

Hence there are two lines possessing the given properties, namely

$$y - 1 = -\frac{1}{7}(x - 3), \text{ or } 7y + x = 10; \quad \text{and} \quad y - 1 = x - 3, \text{ or } y - x = -2.$$

EXERCISES

Group A

1. Find the equation of the line through $(4, -2)$ which is perpendicular to the line $9x + 4y = 6$.

2. Find the equation of the line through the origin which is perpendicular to the line $4x - y = 6$.

3. Determine which of the following lines are parallel to the line $5x - 3y = 12$, which are perpendicular to it, and which are neither.

(*a*) $3y = 5x - 20$ (*d*) $3y = 5(1 - x)$

(*b*) $\frac{x}{5} + \frac{y}{3} = 1$ (*e*) $y = 2 - \frac{5x}{3}$

(*c*) $x = 6y + 5$ (*f*) $3y + 5x = 7$

4. Find the equation of the perpendicular bisector of the line segment joining $(-3, 5)$ and $(1, 7)$.

5. Find the tangent of the obtuse angle between the lines $5x - 3y - 4 = 0$ and $x + 2y = 5$.

6. The vertices of a triangle are $(-4, 1)$, $(1, 5)$ and $(7, 3)$; find the tangents of the angles of the triangle. Construct figure before solving.

7. A line is perpendicular to the line segment connecting $(-3, 8)$ and $(6, 2)$, and cuts that segment one-third of the distance from the former to the latter point; find its equation.

8. Find the equations of the lines making an angle whose tangent is $\frac{1}{4}$ with the line $y = 2x + 3$ at the point $(1, 5)$.

9. A line is perpendicular to $5x + 2y = 0$ and passes through the point $(5, 0)$; find the distance from the origin to the intersection of the two perpendicular lines.

Group B

1. Find the equation of a line through the origin which is perpendicular to the line $4x + 9y = 12$.

2. Find the equation of a line through $(-4, 2)$ which is perpendicular to the line $-4x + y = 6$.

3. Find the equation of the perpendicular bisector of the segment connecting the points $(6, -7)$, $(-3, 5)$.

4. Find the equations of the perpendiculars through the vertices to the opposite sides of the triangle whose vertices are $(-1, 7)$, $(8, -4)$, $(6, -1)$.

5. Find the tangent of the acute angle made by the lines $2x - 3y = 10$ and $x + 3y = 6$.

6. Find the tangents of the angles of the triangle whose vertices are $(-4, -2)$, $(7, -1)$, $(-1, 9)$. Construct figure before solving.

7. The line $Ax + 3y = 6$ makes with the line $2x - y = 4$ an angle whose tangent is $2/3$; find the values of A.

8. What is the value of B if $x - By = 10$ and $3x + 4y = 6$ are perpendicular to each other?

9. Find the equation of a line through the point $(2, 7)$ which with $-x + 4y = 4$ makes an angle whose tangent is $-2/3$. How many such lines are there?

Group C

1. Find the tangent of the obtuse angle made by the lines $3x - 5y = 4$, $2x + 5y = 6$.

2. Find the equation of the perpendicular bisector of the segment whose end-points are $(-7, 4)$, $(8, 2)$.

3. Determine which of the following lines are perpendicular to the line $2x - 7y = 5$, which are parallel to it, and which are neither.

(*a*) $x - \frac{y}{2} = 6$	(*d*) $7y - 2x = 4$
(*b*) $-4x + 14y = -9$	(*e*) $6y + 21x = -4$
(*c*) $7x + 2y = 14$	(*f*) $5x - \frac{7}{4}y = -6$

4. Find the tangents of the angles of the triangle whose sides are $-x + y = 5$, $3x - 7y = 9$, $x + 4y = 20$. Construct figure before solving.

5. Find the equation of a line through the point $(4, -2)$ which is perpendicular to the line $-x + y = 5$.

6. Find the equations of the perpendiculars from the vertices to the opposite sides of the triangle whose vertices are $(-1, 8)$, $(9, 2)$, $(4, 10)$.

7. A line makes an angle whose tangent is $-3/4$ with the line $2x - 5y = 4$; find the slope of the line. Is there more than one such slope?

8. A line perpendicular to the line $2x + 7y - 10 = 0$ passes through the point $(-2, 3)$; find the equation of this line and reduce it to the normal form.

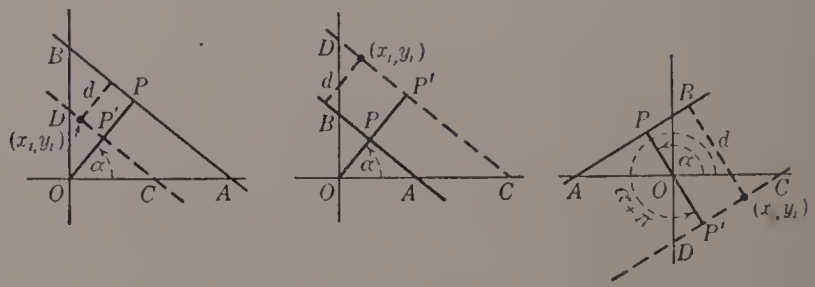
9. The point P divides the segment $A(-2, 4)$, $B(7, 5)$ so that $AP : PB = 7 : 3$; find the equation of the line through P which is perpendicular to AB.

10. Find the value of x if $A(7, 2)$, $B(-2, 7)$, $C(x, 8)$ are the vertices of a triangle having a right angle at C. How many such values of x are there?

40. Distance from a point to a line.—When we seek to represent the distance from a point to a line it is almost inevitable and also highly desirable that distances from all points on one side of the line should be positive and distances from points on the other side of the line should be negative. We are, however, free to arrange matters so that the distances from points on either side we choose shall be positive, when of course the distances from points on the opposite side will be negative.

In the normal form, $x\cos\alpha + y\sin\alpha = p$, of the equation of the straight line, p is the distance from the origin to the line, and we have already agreed to regard this distance as positive in all cases (see §32). To be consistent with this arrangement we will now adopt the following convention.

The distance from a point to a line shall be positive if the point and the origin are on the same side of the line; otherwise the distance shall be negative.



Consider now a line AB whose equation is

$$x\cos\alpha + y\sin\alpha = p,$$

and a point (x_1, y_1) not on this line.

Through (x_1, y_1) pass the line CD parallel to AB.

Suppose that AB and CD are on the same side of the origin (see the first two figures). Then the equation of CD is

$$x\cos\alpha + y\sin\alpha = p',$$

where p' is the distance OP' from the origin to CD (see §32).

Then the distance from (x_1, y_1) to AB is $p - p'$, this distance being positive when (x_1, y_1) is on the same side of the line as the origin, and otherwise negative. Since CD passes through the point (x_1, y_1), $p' = x_1\cos\alpha + y_1\sin\alpha$.

Hence $d = p - p' = p - x_1 \cos \alpha - y_1 \sin \alpha$

which gives the distance from (x_1, y_1) to the line $x \cos \alpha + y \sin \alpha = p$.

If the lines AB and CD are on opposite sides of the origin then $d = p + p'$ (see the last figure).

In this case the equation of CD is

$$x \cos (\alpha + \pi) + y \sin (\alpha + \pi) = p', \text{ (see §32)}$$

and $d = p + p' = p + x_1 \cos (\alpha + \pi) + y_1 \sin (\alpha + \pi)$.

But $\cos (\alpha + \pi) = - \cos \alpha$ and $\sin (\alpha + \pi) = - \sin \alpha$ [see (8), § 251].

Hence in this case also

$$\boldsymbol{d = p - x_1 \cos \alpha - y_1 \sin \alpha} \qquad [13]$$

41. General form of distance formula.—In this section use the figure of §40.

Let the equation of the given line AB be in the general form

$$Ax + By + C = 0. \qquad (1)$$

Then $$\frac{Ax + By}{\pm \sqrt{A^2 + B^2}} = - \frac{C}{\pm \sqrt{A^2 + B^2}} = p$$

in which the sign of $\sqrt{A^2 + B^2}$ is taken so that p is positive.

If, as in §40, a line CD is passed through (x_1, y_1) parallel to AB, then its equation is of the form

$$Ax + By + C' = 0 \qquad (2)$$

and the distance from the origin upon it is

$$\frac{- C'}{\pm \sqrt{A^2 + B^2}} = p'.$$

We now note that if (1) and (2) are on the same side of the origin then C and C' have the same sign and hence in this case the sign of $\sqrt{A^2 + B^2}$ is the same in the values of p and p'.

Hence $$p - p' = - \frac{C}{\pm \sqrt{A^2 + B^2}} + \frac{C'}{\pm \sqrt{A^2 + B^2}} = \frac{- C + C'}{\pm \sqrt{A^2 + B^2}}.$$

But (2) passes through (x_1, y_1) and therefore

$$C' = -Ax_1 - By_1.$$

Hence $$d = p - p' = \frac{- C - Ax_1 - By_1}{\pm \sqrt{A^2 + B^2}},$$

where the sign of $\sqrt{A^2 + B^2}$ is taken so that $\dfrac{- C}{\pm \sqrt{A^2 + B^2}}$ is positive.

Hence $$\boldsymbol{d = \frac{Ax_1 + By_1 + C}{\pm \sqrt{A^2 + B^2}}} \qquad [14]$$

where the sign of $\sqrt{A^2 + B^2}$ is taken so that $\dfrac{C}{\pm \sqrt{A^2 + B^2}}$ is positive.

If the lines (1) and (2) are on opposite sides of the origin, then as in §40, $d = p + p'$, and also C and C' are opposite in sign. Hence $d = p + p'$, reduces to the value given in formula [14].

We therefore have, that for any line $Ax + By + C = 0$ and any point (x_1, y_1),

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

gives the distance from the point to the line provided $\sqrt{A^2 + B^2}$ is given the same sign as that of C.

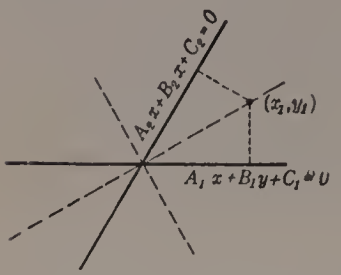
42. Positive and negative sides of a line.—Since in the formula

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}},$$

which gives the distance from (x_1, y_1) to $Ax + By + C = 0$, the sign of $\sqrt{A^2 + B^2}$ is always to be taken the same as that of C, it follows at once that the distance from the origin to the line is positive. It also follows that the distance to a line from any point on the same side of the line as the origin is positive and the distance from any point on the opposite side is negative.

Hence that side of a line on which the origin lies is designated as the **positive side of the line,** and the opposite side as the **negative side.** By this means it may readily be found whether any two points lie on the same or on opposite sides of a line.

43. The bisector of an angle.—Using the distance formula developed in §41, the equations of the bisectors of the angles formed by two intersecting lines may be written directly. It is proved in plane geometry that the locus of all points equally distant from the sides of an angle is the bisector of the angle.



If $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ represent two intersecting lines and if the point (x_1, y_1) is equally distant from these lines, then

$$\frac{A_1x_1 + B_1y_1 + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm\frac{A_2x_1 + B_2y_1 + C_2}{\sqrt{A_2^2 + B_2^2}}.$$

These equations are satisfied by the coördinates of points which are equally distant from the two

lines and by no other values. Note that "distance" is used here as in Euclidean geometry without regard to sign.

Hence the equation of the bisector of one pair of vertical angles formed by the given lines is

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}} \qquad [15]$$

and the equation of the bisector of the other pair of angles is

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = -\frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}} \qquad [15']$$

One of the two lines [15] and [15'] bisects the angle within which the origin lies and its vertical angle, and the other of these two lines bisects the other pair of vertical angles. If in [15] both members reduce to the same sign when $x = 0$, $y = 0$ then this line bisects the angle within which the origin lies and its vertical angle. Otherwise these angles are bisected by [15'].

Example. Find the equations of the bisectors of the pairs of vertical angles formed by the lines $2x - 4y + 7 = 0$, $x + 3y - 9 = 0$, and state which one bisects the angle within which the origin lies.

SOLUTION: From [15] and [15']

$$\frac{x + 3y - 9}{\sqrt{1^2 + 3^2}} = \frac{2x - 4y + 7}{\sqrt{2^2 + 4^2}}, \text{ and } \frac{x + 3y - 9}{\sqrt{1^2 + 3^2}} = -\frac{2x - 4y + 7}{\sqrt{2^2 + 4^2}}$$

are the required bisectors.

Hence, by the remark made above,

$$\frac{x + 3y - 9}{\sqrt{10}} = -\frac{2x - 4y + 7}{2\sqrt{5}} \qquad (1)$$

is the equation of the bisector of the angle within which the origin lies, and of its vertical angle;

and

$$\frac{x + 3y - 9}{\sqrt{10}} = \frac{2x - 4y + 7}{2\sqrt{5}} \qquad (2)$$

is the bisector of the other pair of vertical angles.

Equations (1) and (2) may be reduced to the simpler forms

$$(2 + 2\sqrt{2})x + (6 - 4\sqrt{2})y + 7\sqrt{2} - 18 = 0$$

and

$$(2 - 2\sqrt{2})x + (6 + 4\sqrt{2})y - 7\sqrt{2} - 18 = 0.$$

EXERCISES

1. Find the equations of the bisectors of the angles formed by $-x+8y-3=0$ and $4x-3y+7=0$, and state which one bisects the angle within which the origin lies.

2. Construct the graphs of the two equations $x-3y+8=0$ and $4x-y+7=0$. Discuss the interior of each of the four angles as to its sign with respect to each of these lines. Find the equations of the bisectors and show which angles are bisected by each one.

3. Find the equations of the bisectors of the angles of the triangle whose sides are $5x-y=12$, $2x+4y=9$, $-x+3y=15$. Construct the triangle and note that each bisector must lie in part within the triangle.

44. Equation of line through intersection of two lines.—The equation of a line through the intersection point of two given lines may be found by first solving simultaneously the equations of the given lines, thus obtaining the coördinates of the intersection point. Such an equation may be found more readily, however, by noticing that for all values of k an equation of the form

$$A_1x+B_1y+C_1+k(A_2x+B_2y+C_2)=0 \qquad (1)$$

represents a straight line through the intersection point of $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$.

This is proved as follows: Equation (1) may be reduced to the form

$$(A_1+kA_2)x+(B_1+kB_2)y+C_1+kC_2=0. \qquad (2)$$

Since A_1+kA_2, B_1+kB_2, C_1+kC_2 are constants, (2), and hence also (1), represents a straight line.

Suppose further that (x_1, y_1) is the intersection point of the given lines. Then

$A_1x_1+B_1y_1+C_1=0$ and also $A_2x_1+B_2y_1+C_2=0$ and hence $A_1x_1+B_1y_1+C_1+k(A_2x_1+B_2y_1+C_2)=0$ for all values of k. (Notice that $0+k\cdot 0=0$.)

That is, x_1, y_1 satisfy equation (1) and therefore the point (x_1, y_1) lies on the line (1).

Since line (1) passes through (x_1, y_1) for all values of k, it follows that k may be given a value such as to make the line satisfy any other single condition.

This is illustrated in the following examples.

Example 1. Find the equation of a line through (2, 4) and through the intersection point of $x - 2y + 8 = 0$ and $3x + y - 6 = 0$.

SOLUTION: The general form of the required equation is

$$x - 2y + 8 + k(3x + y - 6) = 0.$$

Since the line passes through (2, 4) we have

$$2 - 8 + 8 + k(6 + 4 - 6) = 0, \text{ and hence } k = -\tfrac{1}{2}.$$

Therefore the equation of the required line is

$$x - 2y + 8 - \tfrac{1}{2}(3x + y - 6) = 0$$

or

$$x + 5y - 22 = 0.$$

Example 2. Find the equation of a line parallel to the line $x - 6y + 3 = 0$ and passing through the intersection point of $-2x + 5y - 12 = 0$ and $3x - 2y + 8 = 0$.

SOLUTION: The equation is of the form

$$-2x + 5y - 12 + k(3x - 2y + 8) = 0,$$

or

$$(3k - 2)x + (5 - 2k)y + 8k - 12 = 0.$$

The slope of this line is $-\dfrac{3k - 2}{5 - 2k}$

while the slope of the line $x - 6y + 3 = 0$ is $\dfrac{1}{6}$.

Hence $-\dfrac{3k - 2}{5 - 2k} = \dfrac{1}{6}$, and $k = \dfrac{7}{16}$.

Therefore $-2x + 5y - 12 + \dfrac{7}{16}(3x - 2y + 8) = 0,$

or

$$11x - 66y + 136 = 0,$$

is the required equation.

EXERCISES

1. Find the equation of the line through the point $(-1, 2)$ which passes through the intersection point of $2x - 4y + 7 = 0$, $x + 3y - 8 = 0$.

2. Find the equation of the line parallel to the line $x + 3y + 6 = 0$ which passes through the intersection of $x - 3y + 4 = 0$, $2x + y - 9 = 0$.

3. Find the equation of the line perpendicular to the line $3x - 4y + 12 = 0$ which passes through the intersection point of $2x - 7y = 9$, $x + 4y = 8$.

4. Discuss the family of lines represented by $x - 3y + 4 + k(2x + 5y - 9) = 0$ as k varies over the real number system.

45. One equation representing two straight lines.—The equations of two straight lines may be combined to form one equation.

Thus, $$(3x - 2y + 4)\,(x + 2y - 6) = 0 \qquad (1)$$

represents the two lines

$$3x - 2y + 4 = 0 \qquad (2)$$

$$x + 2y - 6 = 0. \qquad (3)$$

It is clear that the coördinates of a point on line (2) will satisfy (1) since they make $3x - 2y + 4 = 0$. Similarly the coördinates of a point on line (3) will satisfy (1).

In general $$(A_1x + B_1y + C_1)\,(A_2x + B_2y + C_2) = 0$$

represents the two straight lines

$$A_1x + B_1y + C_1 = 0,\ A_2x + B_2y + C_2 = 0.$$

Example. Form the equation which represents the two straight lines $x + 2y = 3$ and $2x - 5y = 12$.

SOLUTION: It is evident that $(x + 2y - 3)\,(2x - 5y - 12) = 0$ is satisfied by any solution of either of the given equations. Hence we have by multiplying,

$$2x^2 - xy - 10y^2 - 18x - 9y + 36 = 0,$$

which is the required equation.

EXERCISES

1. Write one equation representing the lines $y = 2x + 7$, $2x - y = 3$.

2. Write one equation representing the line $x - 2y = 0$ and the line perpendicular to it at the origin.

3. Find the equation representing the two lines parallel to those of Ex. 2 and passing through the point (6, 3).

4. Construct the graph of the equation $x^2 - y^2 = 0$.

5. For what value of k does $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = k$ represent two straight lines?

6. Write one equation which represents the bisectors of the pairs of vertical angles formed by the lines $3x - 4y = 0$ and $5x - 12y = 0$.

7. Construct the graph of the equation $y(x + 2) = 0$.

EXERCISES

Group A

1. Find the equation of the line through the point $(4, -7)$ which is perpendicular to the line through the points $(-1, 2)$, $(-4, 5)$.

2. Find the equation of the line through the point $(-6, 3)$ which is parallel to the line through the points $(5, 2)$, $(-3, 7)$.

3. Find the equation of a line through the intersection of the lines $x - 7y + 8 = 0$, $3x + 5y - 9 = 0$ which makes an angle of $45°$ with the x-axis.

4. Find the equation of the line through the intersection point of $5x - 4y + 12 = 0$ and $2x + 7y - 10 = 0$ which is perpendicular to the line $7x + 5y - 9 = 0$.

5. Draw the line $x - 4y + 8 = 0$ and point out the positive and negative sides of this line.

6. Draw the line $x + 7y - 14 = 0$, and point out the positive and negative sides of this line.

7. Find the equation of the bisector of that angle formed by $2x - 5y + 8 = 0$ and $3x + 7y - 15 = 0$ within which the origin lies.

8. Find the distance from the intersection of the lines $2x - 5y + 4 = 0$, $x + 3y - 9 = 0$ to the line $x + 5y - 12 = 0$.

9. The points $(2, 1)$, $(7, 5)$, $(3, -6)$ are vertices of a parallelogram; find the fourth vertex. (Three solutions.)

10. Prove that if $\frac{1}{a} + \frac{1}{b} = 6$, then all lines $\frac{x}{a} + \frac{y}{b} = 1$ pass through the point $\left(\frac{1}{6}, \frac{1}{6}\right)$.

Group B

1. Find the equation of the line through the intersection point of $x + y - 9 = 0$, $2x - 3y + 7 = 0$ which is parallel to $4x - 2y + 7 = 0$.

2. Find the equation of the line which passes through the point $(4, 7)$ and is perpendicular to the line through the points $(-6, -3)$, $(2, 5)$.

3. Find the equation of the line whose distance from the origin is 7 and the perpendicular on which from the origin makes an angle of $60°$ with the x-axis.

4. Find the equations of the lines passing through the point $(-3, -6)$ which have equal numerical intercepts on the axes. Note that under the conditions one intercept may be positive and one negative, or both intercepts may be negative. May both intercepts be positive?

5. Draw the line $2x - 3y + 12 = 0$ and point out the positive and negative sides of this line.

6. Draw the line $3x - 5y - 15 = 0$ and point out the positive and negative sides of the line.

7. Find the equation of the bisector of that angle formed by the line $7x - y + 9 = 0$, $3x + 2y - 8 = 0$ within which the origin lies.

8. Find the lines on which lie the three altitudes of the triangle formed by the lines $5x - y + 7 = 0$, $x - 2y + 4 = 0$, $2x + 3y - 10 = 0$.

9. The points $(-7, 0)$, $(2, 3)$, $(5, -3)$ are vertices of a parallelogram; find the fourth vertex. (Three solutions.)

10. Prove that if in $Ax + By - 6 = 0$, $A + B = 6$, then all such equations represent lines which pass through the point $(1, 1)$.

Group C

1. Find the equation of the line through the point $(5, -7)$ which is parallel to the line $2x - 9y + 12 = 0$.

2. Find the tangent of the acute angles formed by the lines $5x - y + 10 = 0$, $x + 3y + 5 = 0$.

3. A line makes an angle whose tangent is $\frac{2}{7}$ with the line $2x + 5y = 10$; find the slope of this line. Is there more than one such line?

4. Find the equation of the perpendicular bisector of the segment connecting the points $(3, 5)$, $(-1, -7)$.

5. Find the equation of the line through $(-1, -3)$ which passes through the intersection point of $-2y + 3x - 9 = 0$, and $2x - 5y + 8 = 0$.

6. Draw the line $5x - 3y + 30 = 0$ and point out the positive and negative sides of the line.

7. Draw the line $3x + 4y - 24 = 0$ and point out the positive and negative sides of the line.

8. Find the equations of the bisectors of the angles formed by the lines $3x + 4y - 24 = 0$, and $5x - 3y + 30 = 0$; which one bisects the angle within which the origin lies?

9. Find the distance from the intersection of the lines $3x - y + 6 = 0$ and $x - 3y - 8 = 0$ to the line $8x - 3y + 12 = 0$.

10. The points $(5, -9)$, $(3, 7)$, $(8, -3)$ are vertices of a parallelogram; find the fourth vertex. (Three solutions.)

MISCELLANEOUS PROBLEMS ON CHAPTER III

1. Find a point on the line $3x - 2y + 8 = 0$ which is equidistant from the points (3, 7), (5, − 2).

SUGGESTION: Let x, y be the coördinates of the point. Then $(x - 3)^2 + (y - 7)^2 = (x - 5)^2 + (y + 2)^2$ and $3x - 2y + 8 = 0$ intersect in the required point.

Solve these equations simultaneously.

2. Find the equation of the line through the point (2, − 5) which divides the segment connecting the points (− 4, 3), (6, − 3) in the ratio of 3 : 2.

3. Prove that if $\frac{1}{a} + \frac{1}{b} = c$ (a constant), then all the lines $\frac{x}{a} + \frac{y}{b} = 1$ pass through a fixed point; find the coördinates of this point. Compare Example 10, page 51.

4. Find the equation of the line through the intersection point of $7x - 3y + 9 = 0$ and $x + 5y - 5 = 0$ which divides the segment connecting the points (3, − 9), (−8, 2) in the ratio 4 : 7.

SUGGESTION: Do not solve the given equations simultaneously.

5. What is the value of B if $2x + By = 16$ passes through the intersection point of $5x - 3y = 12$ and $3y + 2x = 9$?

6. Find m so that $y = mx + 7$ shall bisect the segment connecting the points (− 3, − 7), (5, 2).

7. Prove that all the lines $Ax + By + C = 0$ pass through a common point provided $A + B + C = 0$. Find the point.

SUGGESTION: Substitute $C = - A - B$ obtaining $Ax + By = A + B$.

8. Prove that all the lines $y = mx + b$ pass through a common point provided $m = b$. Find the point.

9. Find the equation of the perpendicular bisector of that part of the line $2x - 9y + 12 = 0$ which lies between its intersection points with the axes.

10. The sides of a triangle are $2x - 9y + 12 = 0$, $5x - 3y + 8 = 0$, $2x + 3y - 9 = 0$; find the equations of the bisectors of the interior angles of this triangle. Construct the triangle before solving.

11. Find the equations of the lines through the point (1, 2) whose distance from the origin is 2.

12. Prove (by the methods of analytic geometry) that the locus of all points equidistant from two points is the perpendicular bisector of the line segment joining the points.

13. Find the equation of the line through the intersection of the lines $5x - 3y + 7 = 0$, $x + 2y + 4 = 0$ which divides the segment $P(2, -4)$, $Q(-5, 10)$ in the ratio $5 : 2$.

14. Find the equation of the line through the intersection of the lines $5x - 9y + 12 = 0$, and $2x + 4y - 7 = 0$ which cuts off equal intercepts on the axes.

15. The point $(5, 7)$ is the vertex of the right angle of a right triangle whose hypotenuse lies in the line $x - 3y - 4 = 0$. A second vertex of the triangle is $(1, -1)$; find the remaining vertex.

16. Find the center and radius of a circle passing through the points $(5, -1)$, $(-2, -2)$, $(6, -8)$

SUGGESTION: Find a point equally distant from these points.

17. The segment connecting the points $(-1, -4)$, $(5, 6)$ is a diagonal of a parallelogram, and the point $(2, -3)$ is an end-point of the other diagonal; find the equation of this second diagonal and its other end-point.

18. Find the equations of the bisectors of the angles of the triangle whose sides are $3x - 4y = 5$, $4x - 3y = 15$, $x + 7y = 35$.

19. A circle with center $(4, -9)$ is tangent to the line $7x - 4y = 12$; find the radius of the circle.

20. A circle with radius 6 is tangent to the line $2x - 5y = 14$ and has its center on the line $x + 4y = 4$; find the center of the circle. How many such circles are there?

21. In the triangle $A(4, 2)$, $B(-5, 3)$, $C(9, -10)$, a median is drawn upon the side BC; find the tangent of the angles which this median makes with the sides AB and AC.

22. Find the points A and B in which the line $x - y = 5$ meets the circle $x^2 + y^2 = 36$; find the equation of the perpendicular bisector of the segment AB and show that it passes through the point $(0, 0)$.

23. Find the equations of the bisectors of the exterior angles of the triangle whose vertices are $(7, 4)$, $(-4, 5)$, $(9, -10)$.

24. Write the equation of a line through the point $(4, 6)$, the normal upon which makes an angle of $120°$ with the x-axis. Construct the figure.

25. Write the equation of a line through the point $(6, 2)$, the normal upon which makes an angle of $300°$ with the x-axis. Construct the figure.

26. Find the value of b if the line $y = x + b$ meets $y^2 = 4x$ in two points whose distance apart is 8.

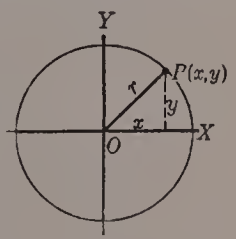
27. Prove that the two bisectors of the angles formed by the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are at right angles to each other.

CHAPTER IV

THE CIRCLE

The word **circle** is used in analytic geometry to mean the locus of all points in a plane which are at a fixed distance from a fixed point. The word does not mean "the part of a plane enclosed by a curve . . . . .", which is the meaning sometimes given to it in Euclidean geometry. When we say, for instance, that a straight line meets a circle in two points the word "circle" must indicate a curve; otherwise the statement is not true.

46. The equation of a circle.—Since $x^2 + y^2$ is the square of the distance from the origin to the point (x, y) it follows that the equation

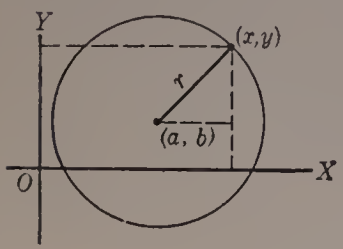


$$x^2 + y^2 = r^2 \qquad [16]$$

holds for all points (x, y) whose distance from the origin is r. Hence [16] is the equation of a circle with radius r and center at the origin.

The square of the distance between the points (a, b) and (x, y) is $(x - a)^2 + (y - b)^2$. Hence the equation

$$(x - a)^2 + (y - b)^2 = r^2 \qquad [17]$$



is the equation of a circle with radius r and center (a, b).

In this equation a, b, and r are fixed; that is, the center and the radius are fixed and x and y are allowed to vary subject to the condition stated in the equation. The point (x, y) may then take any position on the circle, but cannot be either inside or outside the circle.

SIGHT WORK

Write the equations of circles if:

(*a*) center is (0, 0), $r = 6$; (*b*) center is $(-2, 3)$, $r = 4$;
(*c*) center is $(3, -5)$, $r = 5$; (*d*) center is $(-6, -3)$, $r = 7$.

47. General form of the equation of a circle.—The equation

$$(x - a)^2 + (y - b)^2 = r^2$$

reduces to

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0,$$

which is of the general form

$$x^2 + y^2 + 2Dx + 2Ey + C = 0. \tag{1}$$

We will now show that in case an equation of the general form (1) represents a real locus, then this locus is a circle.

Transposing C and completing the squares in $x^2 + 2Dx$ and $y^2 + 2Ey$ by adding D^2 and E^2,

we have $x^2 + 2Dx + D^2 + y^2 + 2Ey + E^2 = D^2 + E^2 - C,$

or $(x + D)^2 + (y + E)^2 = D^2 + E^2 - C.$

If we represent $D^2 + E^2 - C$ by r^2, the equation is

$$(x + D)^2 + (y + E)^2 = r^2,$$

which is the equation of a circle with center $(-D, -E)$ and radius r.

Since r^2 is positive, $D^2 + E^2 - C$ must be positive in order that (1) shall represent a circle.

Note that an equation of the general form

$$Ax^2 + Ay^2 + 2Dx + 2Ey + C = 0 \tag{2}$$

may be reduced to the form of equation (1) by dividing by A. The point is that the coefficients of x^2 and y^2 must be equal.

Hence (2) is the equation of a circle whose center is $\left(-\frac{D}{A}, -\frac{E}{A}\right)$ and the square of whose radius is $\left(\frac{D}{A}\right)^2 + \left(\frac{E}{A}\right)^2 - \frac{C}{A}$

Example 1. Find the center and radius of the circle $x^2 + y^2 - 4x + 6y - 8 = 0$.

SOLUTION: In this case $D = -2$, and $E = 3$. Hence $(2, -3)$ is the center of the circle.

$$D^2 + E^2 - C = 4 + 9 + 8 = 21 = r^2. \quad \text{Hence } r = \sqrt{21}.$$

Example 2. Find the center and radius of the circle $x^2 + y^2 + 2x - 4y + 20 = 0$.

SOLUTION: The center $(-D, -E)$ is $(-1, 2)$ and $D^2 + E^2 - C = 1 + 4 - 20 = -15$. Hence the equation has no locus.

Example 3. Find the center and radius of $x^2 + y^2 - 4x + 8y + 20 = 0$.

SOLUTION: The center $(-D, -E)$ is $(2, -4)$ and $D^2 + E^2 - C = 4 + 16 - 20 = 0$; hence the radius is zero and the locus is the point $(2, -4)$.

Example 4. Find the center and radius of $3x^2 + 3y^2 - 6x + 4y - 10 = 0$.

SOLUTION: Dividing by 3 gives $x^2 + y^2 - 2x + \frac{4}{3}y - \frac{10}{3} = 0$.

Hence $(1, -\frac{2}{3})$ is the center.

Also, $(1)^2 + \left(-\frac{2}{3}\right)^2 + \frac{10}{3} = 1 + \frac{4}{9} + \frac{10}{3} = \frac{43}{9}$, and $r = \frac{1}{3}\sqrt{43}$.

EXERCISES

1. Find the center and radius of each of the following circles:
   (*a*) $x^2 + y^2 - 6x - 8y + 9 = 0$.
   (*b*) $4x^2 + 4y^2 + 4x - 10y - 8 = 0$.
   (*c*) $x^2 + y^2 + 8x + 10y + 25 = 0$.
   (*d*) $2x^2 + 2y^2 + 5x - 7y - 10 = 0$.
2. Give the equation of the circle of radius r,
   (*a*) with center on the x-axis at a distance a from the origin,
   (*b*) with center on the y-axis at a distance b from the origin,
   (*c*) with center on the x-axis and tangent to the y-axis,
   (*d*) with center on the y-axis and tangent to the x-axis,
   (*e*) in the first quadrant and tangent to both axes.
3. A circle whose center is $(5, 6)$ passes through the point $(-1, 3)$; find its equation.
4. Find the equation of a circle of radius 6 which is tangent to the y-axis and has 4 as the ordinate of its center. How many such circles are there?

5. Find the equation of a circle of radius 10 which is tangent to the y-axis and which has its center on the line $y = 2x$. How many such circles are there?

6. Find the points of intersection of the line $x = 2y + 1$ with the circle $x^2 + y^2 - 2x - 19 = 0$.

7. A circle with radius 5 has its center on the line $x - y = 1$ and passes through the origin; find its equation. How many such circles are there?

SUGGESTION: The center must lie upon a circle of radius 5 with center at the origin.

8. Find the points of intersection of the line $3x - 4y + 35 = 0$ and the circle $x^2 + y^2 = 49$. Construct a graph showing the relation of the line to the curve.

9. Find the points of intersection of the circles $x^2 + y^2 = 100$ and $x^2 - 16x + y^2 - 16y + 124 = 0$. Write the equation of the line through these two points.

10. Find the equation of the circle whose center is at the origin and which is tangent to the line $4y = 3x + 10$.

48. Circle determined by three points.—It is found in plane geometry that a circle may be passed through three points provided they do not lie in a straight line. We will now show how to find the equation of the circle through three given points.

Example. Find the equation of the circle through the points (1, 2), (− 2, 4), (3, − 3).

If the locus represented by the equation

$$x^2 + y^2 + 2Dx + 2Ey + C = 0 \tag{1}$$

passes through these points, then

$$\begin{aligned} 1^2 + 2^2 + 2D(1) + 2E(2) + C &= 0 \\ (-2)^2 + 4^2 + 2D(-2) + 2E(4) + C &= 0 \\ 3^2 + (-3)^2 + 2D(3) + 2E(-3) + C &= 0 \end{aligned}$$

or

$$\begin{cases} 5 + 2D + 4E + C = 0 \\ 20 - 4D + 8E + C = 0 \\ 18 + 6D - 6E + C = 0. \end{cases} \tag{2}$$

Solving these equations for D, E, C and substituting in (1) we have

$$x^2 + y^2 + \frac{101}{11}x + \frac{69}{11}y - \frac{294}{11} = 0,$$

or $$11x^2 + 11y^2 + 101x + 69y - 294 = 0,$$

which is the required equation.

This problem may also be solved by finding the equations of the perpendicular bisectors of the segments $A(1, 2)$, $B(-2, 4)$; $B(-2, 4)$, $C(3, -3)$ and solving them simultaneously to find the center of the circle.

That is, $\frac{4-2}{-2-1} = -\frac{2}{3}$ is the slope of AB and $\left(-\frac{1}{2}, 3\right)$ is its middle point. Hence $y - 3 = \frac{3}{2}\left(x + \frac{1}{2}\right)$ or $6x - 4y + 15 = 0$ (1) is the perpendicular bisector of AB. In the same manner we find that $5x - 7y + 1 = 0$ is the perpendicular bisector of BC.

Solving (1) and (2), $x = -\frac{101}{22}$ and $y = -\frac{69}{22}$, which are the coördinates of the center of the circle.

The distance between (1, 2) and the center is found to be $\sqrt{\frac{27898}{22^2}}$. Then the equation of the circle is

$$\left(x + \frac{101}{22}\right)^2 + \left(y + \frac{69}{22}\right)^2 = \frac{27898}{22^2},$$

which reduces to $11x^2 + 101x + 11y^2 + 69y - 294 = 0$.

If the given points lie in a straight line the resulting set of equations (2) will be contradictory and the solution becomes impossible. If the second method is used the two perpendicular bisectors will then be parallel. It should be noticed that the number of points is the same as the number of constants in equation (1); and hence the set of equations (2) contains as many equations as there are unknowns.

SIGHT WORK

1. Using the first method shown above find the equation of the circle through the points (0, 0), (4, 7), (3, 0); also find this circle using the second method.

2. Find the equation of the circle through the points $(-1, -1)$, $(2, -1)$, (4, 6), using either of the methods shown above.

49. Definition of tangent to a circle.—A tangent to a circle may be defined as a line which meets the circle in only one point, no matter how far the line is extended. This is the definition used in plane geometry and will be used here. In §123 a more general definition of tangent to a curve is given.

50. Tangents with a given slope.—The equation of the tangent with a given slope m may be written in the form

$$y = mx + b. \tag{1}$$

Consider the circle

$$x^2 + y^2 = r^2. \tag{2}$$

To make the line (1) tangent to this circle it is necessary to find values of b which will make it meet (2) in only one point.

Substituting from (1) in (2) we have,

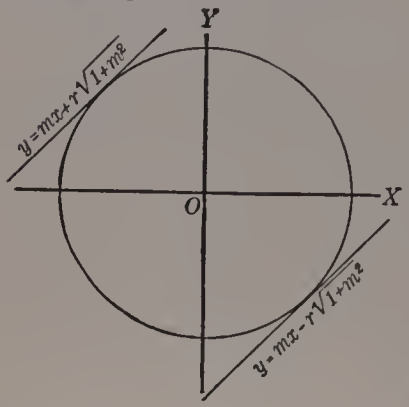
$$x^2 + (mx + b)^2 = r^2,$$

or
$$x^2(1 + m^2) + 2bmx + b^2 - r^2 = 0.$$

Hence,
$$x = \frac{-bm \pm \sqrt{r^2(1 + m^2) - b^2}}{(1 + m^2)}$$

are the abscissas of the two points in which the line meets the circle.

If the line is to meet the circle in only one point, these two values of x must be equal. That is, the expression under the radical sign must be zero.



Hence, $r^2(1 + m^2) - b^2 = 0$

and $b = \pm r\sqrt{1 + m^2}$.

Substituting these values of b in (1) we have,

$$y = mx + r\sqrt{1 + m^2}$$

and $$y = mx - r\sqrt{1 + m^2} \quad [18]$$

It is clear from the figure that a circle will have two tangents with a given slope. These are the two lines given by the equations [18].

Note that either of these equations represents a tangent to the circle, no matter what the value of m is.

All tangents except those perpendicular to the x-axis may be represented by this form of the equation.

A method for finding the tangents with a given slope when the equation of the circle is in the form

$$x^2 + y^2 + 2Dx + 2Ey + C = 0$$

is shown in the following:

Example. Find the equations of two lines with slope $\frac{2}{3}$ which are tangent to the circle $x^2 + y^2 - 4x + 6y - 10 = 0$.

SOLUTION: Solve simultaneously $x^2 + y^2 - 4x + 6y - 10 = 0$ and $y = \frac{2}{3}x + b$, and find the values of b which reduce the radical to zero.

The result is $$b = \frac{-13 \pm \sqrt{299}}{3}.$$

Hence, $$y = \frac{2}{3}x + \frac{-13 + \sqrt{299}}{3}, \quad y = \frac{2}{3}x + \frac{-13 - \sqrt{299}}{3}$$

are the two tangents required.

EXERCISES

1. Find the equations of the tangents to $x^2 + y^2 = 36$ if the slope of the tangents is $\frac{2}{5}$.

2. Find the equations of the tangents to the circle $x^2 + y^2 = 25$ which are parallel to the line $2x - 3y + 7 = 0$.

3. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ which are perpendicular to the line $x - 2y + 4 = 0$.

4. Find the equations of the tangents to the circle $x^2 + y^2 = 64$ which are parallel to the line joining the points $(a, 0)$, $(0, b)$.

5. Find the equations of the tangents to the circle $x^2 + y^2 = 49$ which are perpendicular to the line joining the points $(2, -1)$, $(-3, 7)$.

6. Find the equations of the tangents to the circle $x^2 + y^2 + 4x - 6y + 5 = 0$ which are parallel to the line $x + y = 4$.

7. Find the equations of the tangents to the circle $x^2 + y^2 - 6x + 4y + 12 = 0$ which are perpendicular to the line $2x + y = 8$.

51. Tangent to a circle at a point on it.—To find the equation of the tangent to a circle through a given point on it we make use of the proposition that a line perpendicular to a radius at its extremity is tangent to the circle.

Let (x_1, y_1) be a point on the circle $x^2 + y^2 = r^2$. The slope of the radius to the point (x_1, y_1) is $\frac{y_1}{x_1}$ and hence the slope of the tangent at this point is $-\frac{x_1}{y_1}$ (see §38). The equation of the tangent is therefore

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1),$$

or

$$x_1x + y_1y = x_1^2 + y_1^2.$$

But the point (x_1, y_1) lies on the circle and hence

$$x_1^2 + y_1^2 = r^2.$$

Therefore,

$$\boldsymbol{x_1x + y_1y = r^2} \qquad [19]$$

is the equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1).

Again let (x_1, y_1) be a point on the circle $x^2 + y^2 + 2Dx + 2Ey + C = 0$.

The center of the circle is $(-D, -E)$, and the slope of the radius to the point (x_1, y_1) is

$$\frac{y_1 + E}{x_1 + D}.$$

Hence the slope of the tangent is $-\frac{x_1 + D}{y_1 + E}$ and its equation is

$$y - y_1 = -\frac{x_1 + D}{y_1 + E}(x - x_1),$$

or

$$x_1x + y_1y + Dx + Ey = x_1^2 + y_1^2 + Dx_1 + Ey_1.$$

Adding $Dx_1 + Ey_1$ to both members we have

$$x_1x + y_1y + D(x + x_1) + E(y + y_1) = x_1^2 + y_1^2 + 2Dx_1 + 2Ey_1.$$

But (x_1, y_1) lies on the circle and hence

$$x_1^2 + y_1^2 + 2Dx_1 + 2Ey_1 + C = 0, \quad \text{or} \quad x_1^2 + y_1^2 + 2Dx_1 + 2Ey_1 = -C.$$

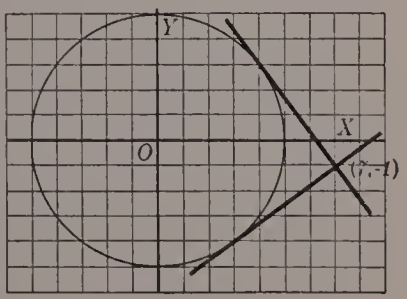
Therefore, $\boldsymbol{x_1x + y_1y + D(x + x_1) + E(y + y_1) + C = 0}$ [20]

which is the required equation.

Compare equations [19] and [20]. Show that if $D = 0$, and $E = 0$ in [20], we have equation [19]. In this case C is negative and $-C = r^2$.

52. Tangents to a circle through a given point outside the circle.—General formulas are not given here, but methods for finding the equations of such tangents are shown by two examples.

Example 1. Find the equations of lines through $(7, -1)$ which are tangent to the circle $x^2 + y^2 = 25$.



SOLUTION: The line through $(7, -1)$ with slope m is

$$y + 1 = m\,(x - 7). \qquad (1)$$

The distance from the center of the given circle $(0, 0)$ to this line is, by §41,

$$\frac{7m + 1}{\sqrt{1 + m^2}}.$$

Then $$\frac{7m + 1}{\sqrt{1 + m^2}} = 5,$$

and hence $$49m^2 + 14m + 1 = 25 + 25m^2.$$

Solving for m, $$m = -\frac{4}{3} \text{ or } \frac{3}{4}.$$

Substituting in (1) we have

$$y + 1 = -\frac{4}{3}(x - 7) \text{ and } y + 1 = \frac{3}{4}(x - 7),$$

which reduce to

$$4x + 3y = 25 \text{ and } 3x - 4y = 25.$$

These are the required equations.

Example 2. Find the equations of the tangents to the circle $x^2 + y^2 - 8x + 2y - 6 = 0$, which pass through the point $(10, 0)$.

SUGGESTION FOR SOLUTION: The general equation of a line through the point $(10, 0)$ is

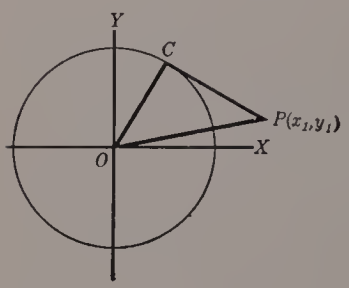
$$y = m(x - 10).$$

Solve this equation simultaneously with

$$x^2 + y^2 - 8x + 2y - 6 = 0,$$

and find the values of m for which the line meets the circle in only one point. Substituting these values of m in $y = m(x - 10)$ will give the required equations.

53. Length of tangent from a point to a circle.—Let $P(x_1, y_1)$ be a point outside the circle $x^2 + y^2 = r^2$. Then the square of the distance from P to the origin O is $x_1^2 + y_1^2$. Let C be the point of contact of one of the tangents. Then PCO is a right triangle and



$$\overline{PO}^2 = \overline{PC}^2 + \overline{OC}^2,$$

or $$PC = \sqrt{\overline{PO}^2 - \overline{OC}^2}.$$

But $$\overline{OC}^2 = r^2.$$

Hence, $$PC = \sqrt{x_1^2 + y_1^2 - r^2}.$$

Again, let $P(x_1, y_1)$ be a point outside the circle

$$x^2 + y^2 + 2Dx + 2Ey + C = 0.$$

Then the square of the distance from the center to (x_1, y_1) is $(x_1 + D)^2 + (y_1 + E)^2$ and the square of the radius of the circle is $D^2 + E^2 - C$ (see §47).

Hence the square of the length of the tangent is

$$\begin{aligned} t^2 &= (x_1 + D)^2 + (y_1 + E)^2 - D^2 - E^2 + C \\ &= x_1^2 + y_1^2 + 2Dx_1 + 2Ey_1 + C, \end{aligned} \tag{1}$$

and we have:

The square of the length of a tangent from a point to a circle

$$x^2 + y^2 + 2Dx + 2Ey + C = 0$$

is obtained by substituting the coördinates of the point in the left member of the equation.

Obviously this result holds for the circle $x^2 + y^2 = r^2$ when written in the form $x^2 + y^2 - r^2 = 0$.

This result also gives a convenient method for determining whether a point is outside or inside a given circle. If the result given in (1) is positive, the point (x_1, y_1) is outside the circle; if the result is negative, the point is inside the circle, since in that case the length of the tangent would be imaginary. Obviously, if the result is zero, the point is on the circle.

EXERCISES

Group A

1. Find the center and radius of the circle $x^2 + y^2 - 8x + 4y - 12 = 0$.

2. Find the equation of the circle whose center is $(-2, 3)$ if it passes through the point $(5, -4)$.

3. Find the equation of the circle whose center is $(5, -3)$ if it is tangent to the line $2x - 3y + 4 = 0$.

4. Find the equation of the circle whose center lies on the line $x - 4y + 6 = 0$ if it passes through the two points $(5, -3)$ and $(-3, -7)$.

5. Find the equation of the circle which passes through the three points $(5, -5)$, $(1, -3)$, $(4, -6)$.

6. Find the equation of the circle whose center is $(5, 3)$ and the length of a tangent to which from the point $(-3, -7)$ is 5.

7. Find the equations of the tangents to the circle $x^2 + y^2 = 25$ whose slopes are $3/7$.

8. Find the equation of the tangent to the circle $x^2 + y^2 = 100$ which passes through the point $(6, 8)$.

9. Find the equations of the tangents to the circle $x^2 + y^2 = 25$ which pass through the point $(6\frac{1}{4}, 0)$.

10. Find the equation of the tangent to the circle $x^2 + y^2 + 4x - 10y - 24 = 0$ which passes through the point $(5, 7)$ on it.

11. Find the values of b which make the line $y = 2x + b$ tangent to the circle $x^2 + y^2 - 3x + 5y + 8 = 0$.

12. Decide which of the points $(1, 1)$, $(5, 3)$, $(7, 8)$, $(-1, -10)$ lie within the circle $x^2 + y^2 - 4x + 6y - 6 = 0$ and which lie outside it.

Group B

1. Find the equation of the circle whose center is $(1, -9)$ and which is tangent to the line $x - y + 8 = 0$.

2. Find the center and radius of the circle $x^2 + y^2 - 2x - 9y + 4 = 0$.

3. Find the equation of the circle whose center is $(1, 7)$ if it passes through the point $(-1, -1)$.

4. Find the equation of the circle whose center lies on $5x - y + 6 = 0$ if it passes through the points $(5, -7)$, $(1, -3)$.

5. Find the equation of the circle which passes through the three points $(6, 2)$, $(9, -3)$, $(4, -6)$.

6. Find the equation of the circle whose center is $(1, 4)$ and the length of a tangent to which from the point $(5, 7)$ is 2.

7. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ whose slopes are $-\frac{3}{4}$.

8. Find the equation of the tangent to the circle $x^2 + y^2 = 64$ which passes through the point $(4, 4\sqrt{3})$.

9. Find the equations of the tangents to the circle $x^2 + y^2 = 25$ which pass through the point $(0, 8\frac{1}{3})$.

10. Find the equation of the tangent to the circle $x^2 + y^2 - 10x + 6y - 2 = 0$ which passes through the point $(5, 3)$ on it.

11. Find the values of m for which the line $y = mx + 10$ is tangent to the circle $x^2 + y^2 = 49$.

12. Find the values of m for which the line $y = mx + 8$ is tangent to the circle $x^2 + y^2 - 4x + 2y - 8 = 0$.

Group C

1. Find the equation of the circle whose center lies on the x-axis if it passes through the points $(2, 1)$, $(3, -1)$.

2. Find the equation of the circle whose center is $(2, 3)$ if it passes through the point $(5, 7)$.

3. Find the equation of the circle whose center is $(-1, -6)$ if it is tangent to the line $x - 3y + 6 = 0$.

4. Find the equation of the circle whose center lies on $4x + 13y + 2 = 0$ if it passes through the points $(5, 3)$, $(1, -3)$.

5. Find the equation of the circle which passes through the points $(9, 3)$, $(3, -5)$, $(10, -4)$.

6. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ which pass through the point $(6, -4)$.

7. Find the equation of the tangent to the circle $x^2 + y^2 = 49$ which passes through the point $(2\sqrt{10}, 3)$.

8. Find the values of m which make the line $y = mx + 8$ tangent to the circle $x^2 + y^2 = 36$.

9. Find the values of m which make the line $y = mx + 7$ tangent to the circle $x^2 + y^2 - 2x + 2y - 6 = 0$.

10. Decide which of the points $(1, 4)$, $(-1, -2)$, $(4, -7)$, $(8, -2)$, lie within the circle $x^2 + y^2 - 2x + 9y - 10 = 0$ and which lie outside it.

11. Find the equation of the circle whose center is $(1, 3)$ and the length of a tangent to which from the point $(8, -6)$ is 3.

12. Find the equation of the tangent to the circle $x^2 + y^2 - 8x + 4y - 48 = 0$ which passes through the point $(2, 6)$ on it.

54. Family of circles through the intersection of two given circles.—In §44 we found that for all values of k the equation

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0 \qquad (1)$$

represents a straight line through the intersection point of the lines $A_1x + B_1y + C_1 = 0$, and $A_2x + B_2y + C_2 = 0$.

We shall now find in a similar manner that for all values of k the equation

$$x^2 + y^2 + 2D_1x + 2E_1y + C_1 + k(x^2 + y^2 + 2D_2x + 2E_2y + C_2) = 0 \qquad (2)$$

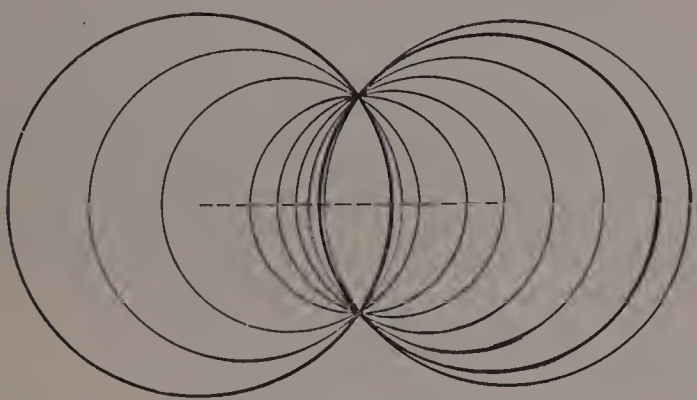
represents a straight line or a circle through the two intersection points of the circles

$$x^2 + y^2 + 2D_1x + 2E_1y + C_1 = 0 \qquad (3)$$

$$x^2 + y^2 + 2D_2x + 2E_2y + C_2 = 0. \qquad (4)$$

By the argument used in §44 equation (2) is satisfied by the coördinates of any point which lies on the locus of both (3) and (4). That is, the locus of (2) passes through the intersection points of (3) and (4).

Hence it remains to show that (2) represents a straight line or a circle.



Equation (2) reduces to

$$(1 + k)x^2 + (1 + k)y^2 + (2D_1 + 2kD_2)x + (2E_1 + 2kE_2)y + C_1 + kC_2 = 0. \qquad (5)$$

If $k = -1$, this reduces to a first degree equation in x and y and hence represents a straight line.

If k is not equal to -1, then we notice that the coefficients of x^2 and y^2 are equal and (5) represents a circle (see §47) for all such values of k different from -1.

55. Radical axis of two circles.—We have just seen that in case $k = -1$, then

$$x^2 + y^2 + 2D_1x + 2E_1y + C_1 + k(x^2 + y^2 + 2D_2x + 2E_2y + C_2) = 0$$

reduces to a straight line.

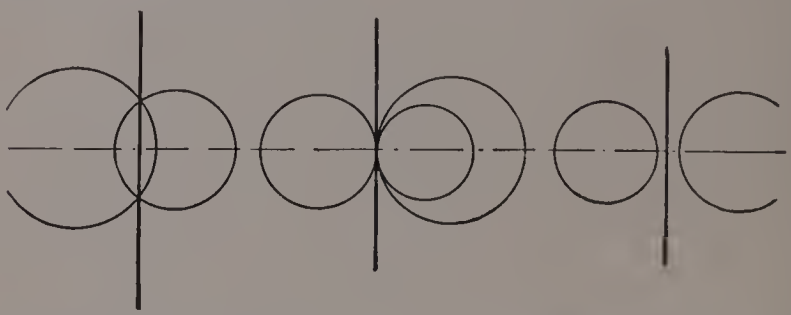
This line, $$2(D_1 - D_2)x + 2(E_1 - E_2)y + C_1 - C_2 = 0, \quad (1)$$

is called the **radical axis** of the circles

$$x^2 + y^2 + 2D_1x + 2E_1y + C_1 = 0 \quad (2)$$

$$x^2 + y^2 + 2D_2x + 2E_2y + C_2 = 0. \quad (3)$$

That is, if the equations of two circles are transformed so that the coefficients of x^2 and y^2 are unity, as in (2) and (3), then the equation of the radical axis is obtained by subtracting (3) from (2). If the circles meet in two points the radical axis passes through these points; if the circles are tangent one to the other, the radical axis is tangent to both circles at their common point of tangency; and if the circles do not meet, the radical axis passes between them. See §188.



In plane geometry it is shown that if a line is drawn through the intersection points of two circles and if from a point on this line tangents are drawn to the two circles, then the tangents are equal in length. In §188 of this book it is shown that the tangents to two circles drawn from any point on their radical axis are equal in length. This is true whether the radical axis meets the circles or not.

EXERCISES

Group A

1. Find the equation of the circle passing through the points (2, 2), (4, − 2), (8, 0). Use the method suggested on page 59.

2. Find the equation of the circle which passes through the point (3, 5) and through the intersection points of $x^2 + y^2 - 8x + 10y - 12 = 0$ and $x^2 + y^2 + 10x - 8y - 12 = 0$.

SUGGESTION: From §54 we know that for all values of k, $x^2 + y^2 - 8x + 10y - 12 + k(x^2 + y^2 + 10x - 8y - 12) = 0$ passes through the intersection points of the given circles. Find the value of k which will make this locus pass through the point (3, 5).

3. Find the locus of the centers of circles passing through the intersections of $x^2 + y^2 - 6x + 10y + 4 = 0$, $x^2 + y^2 + 10x - 4y - 20 = 0$.

SUGGESTION: Find the equation of the line passing through the centers of the two circles. What theorem from plane geometry is used?

4. Find the equation of the circle through (8, 9) which is tangent to the two axes.

5. Find the equations of circles through the points (4, 6), (8, − 2) if the radius is 5.

6. Find the equation of the circle through the intersection points of $x^2 + y^2 + 4x - 8y + 4 = 0$, and $x^2 + y^2 - 12x + 6y - 12 = 0$ if it passes through (1, − 3).

7. A circle with radius 5 passes through the point (− 3, − 1) and has its center on the line $y = x + 1$; find the equation. How many such circles are there?

8. Find the values of x if the circle through the points (1, 0), (− 2, 0), (− 3, 6) passes through the point $(x, 5)$.

Group B

1. Find the equation of the circle through the points (− 2, − 4), (8, 2) if its center is on the line $5x - 3y + 12 = 0$.

2. Find the equation of the circle which passes through the points (4, − 1), (−2, − 5), (3, − 6).

3. A circle with radius 8 has its center on the line $x - y - 1 = 0$, and passes through the origin; find its equation. How many such circles are there?

4. Find the equation of the circle through the intersection points of $x^2 + y^2 - 4x + 5y - 8 = 0$, $x^2 + y^2 + 6x - 4y - 6 = 0$ whose center is on the line $y + x = 1$.

5. Prove that the radical axis of two circles is perpendicular to their line of centers. Does this hold when the circles do not meet?

Suggestion: Let $x^2 + y^2 = r_1^2$, and $(x - a)^2 + y^2 = r_2^2$ be the equations of the circles. Does this arrangement make the proof general so as to apply to any two circles?

6. Find the value of k if the circle

$$x^2 + y^2 - 10x + 5y - 10 + k(x^2 + y^2 - 2x + 4y + 10) = 0$$

passes through the point (2, 7).

7. Find the values of y if the point $(2, y)$ is on the circle through the points (2, 5), (−1, 0), (− 6, 3).

8. Find the equation of a circle which passes through the point (− 4, 2) and is tangent to the coördinate axes. How many such circles are there?

Group C

1. Find the equation of the circle passing through the points (0, 1), (5, 0), (3, 6). Use the method shown on page 58.

2. Find the equation of the circle which passes through the points (5, 6), (6, 5) if its radius is 5. How many such circles are there?

3. A circle with radius 2 has its center on the line $x + y = 5$ and passes through the point (2, 1); find the equation of the circle. How many such equations are there?

4. Find the locus of the centers of the circles which pass through the intersection points of the circles $x^2 + y^2 - 4x + 2y - 6 = 0$, $x^2 + y^2 + 6x - 4y - 10 = 0$. See *Suggestion* under Ex. 3, p. 69.

5. Find the equation of the circle through (− 4, − 2) which is tangent to the two axes. How many such circles are there?

6. A circle has its center on the line $x + y = -3$ and passes through the intersections of the circles, $x^2 + y^2 - 4x + 2y - 6 = 0$, $x^2 + y^2 + 6x - 4y - 10 = 0$; find the equation of the circle.

7. Find the equation of a circle which passes through the point (8, − 4) and has its center at (− 1, 2).

8. A circle is tangent to the x-axis and passes through the intersections of the circles $x^2 + y^2 - 8x - 10y + 12 = 0$ and $x^2 + y^2 - 6x - 12y + 8 = 0$; find its equation. How many such circles are there?

9. Find the values of k for which the circle $x^2 + y^2 - 8x - 10y + k(x^2 + y^2 - 6x - 12y + 8) = 0$ is tangent to the x-axis.

MISCELLANEOUS PROBLEMS ON CHAPTER IV

1. Lines are drawn from the point (3, 2) tangent to the circle $x^2 + y^2 = 1$; find the slopes of the tangents.

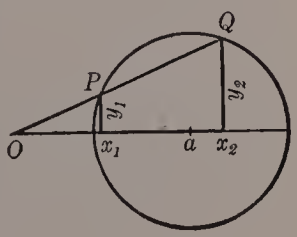
2. Lines with slope -1 are tangent to the circle $x^2 + y^2 - 8x - 12y + 27 = 0$; find the equations of the lines.

3. Find the equation of a circle inscribed in a triangle whose sides are $x = 0$, $y = 0$, and $5x + 12y = 13$.

4. Find the square of the length of a tangent from $P(4, 6)$ to the circle $x^2 + y^2 = 16$.

5. Find the square of the length of the tangents from the origin to the circle $(x - a)^2 + y^2 = r^2$; also find the product of the distances from the origin to the points where the circle cuts the x-axis, and compare.

6. Prove analytically: If lines are drawn from a fixed point O meeting a circle in P and Q, then the product $OP \cdot OQ$ is constant for all such lines.



SUGGESTION: Let the fixed point be the origin and let the equation of the circle be $(x - a)^2 + y^2 = r^2$. Then the equation of the line is $y = mx$. Substituting mx for y in the equation of the circle,

$$x^2(1 + m^2) - 2ax + a^2 - r^2 = 0.$$

Hence by §241 the product of the roots x_1 and x_2 of this equation is $x_1x_2 = \dfrac{a^2 - r^2}{1 + m^2}$.

Using $m = \dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$ show that $OP = x_1\sqrt{1 + m^2}$, $OQ = x_2\sqrt{1 + m^2}$, and hence that $OP \cdot OQ = a^2 - r^2$. (Compare Example 2, page 145.)

7. Find the distance from the point (5, 4) to the nearest point of the circle $x^2 + y^2 + 14x + 2y + 14 = 0$.

SUGGESTION: First find the distance from the point (5, 4) to the center of the circle.

8. Find the locus of the centers of the circles which pass through the intersection points of $x^2 + y^2 = 25$ and $x^2 + y^2 - 10x - 10y + 45 = 0$.

9. Prove that the locus of the centers of all circles which pass through the intersection points of two given circles is a straight line.

SUGGESTION: Let $x^2 + y^2 = r_1^2$ and $(x - a)^2 + y^2 = r_2^2$ be the equations of the circles.

10. Find the equation of the circle circumscribed about the triangle whose sides are $x + y + 3 = 0$, $2x - y + 6 = 0$, $x - y + 6 = 0$.

11. Find the equation of a circle tangent to the line $2x - 4y + 7 = 0$ which is concentric with the circle $3x^2 + 3y^2 - 10x + 12y - 18 = 0$.

12. Find the locus of a point which moves so that the sum of the squares of the distances from the sides of an equilateral triangle is constant.

SUGGESTION: Let $(-a, 0)$, $(a, 0)$ be the extremities of one side of the triangle.

13. Find the locus of a point which moves so that the sum of the squares of its distances from the sides of a rectangle is constant.

14. A point moves so that its distance from one fixed point is in a constant ratio (but not equal) to its distance from a second point; show that its locus is a circle.

15. A circle is tangent to the lines $3x + y + 5 = 0$, $x + 3y + 3 = 0$ and has its center on $5x - 4y - 8 = 0$; find its equation.

16. The tangents to the circle $x^2 + y^2 = 36$ from the point P are at right angles to each other; find the locus of P.

17. The tangents to a circle from a point P make a fixed angle with each other; find the locus of P.

SUGGESTION: Let $x^2 + y^2 = r^2$ be the equation of the circle.

18. Find the locus of the point P if the sum of the squares of its distances from the points $(-a, 0)$, $(a, 0)$ is $4a^2$.

19. Find the locus of the point P if the sum of the squares of its distances from two fixed points is constant.

20. Find the locus of the point P if the sum of the squares of its distances from n fixed points is constant.

21. Show analytically that an angle inscribed in a semicircle is a right angle.

SUGGESTION: Let the segment joining $(-a, 0)$, $(a, 0)$ be the diameter of the semicircle.

22. Find the locus of the vertex of a triangle with a fixed base if it moves so that the sum of the base angles of the triangle is constant.

23. Find the locus of a point which moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides. Show that the locus passes through the extremities of the base of the triangle.

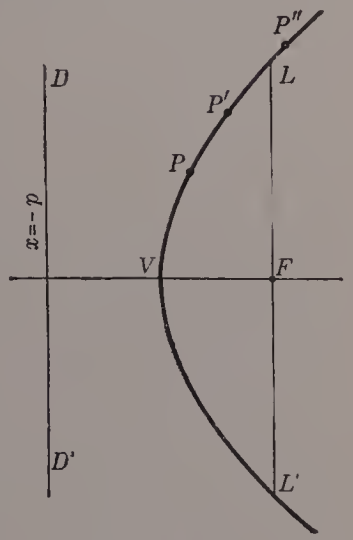
SUGGESTION: Let $(-a, 0)$, $(a, 0)$, $(0, b)$ be the vertices of the triangle.

CHAPTER V

THE PARABOLA

A **parabola** is the locus of all points whose distances from a fixed point and from a fixed line are equal.

The fixed line is the **directrix** of the parabola and the fixed point is its **focus.**

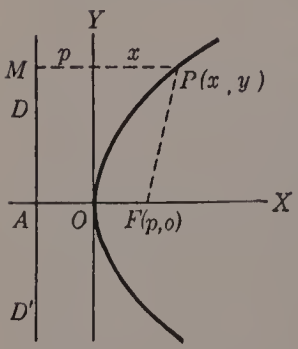


In the figure let $D'D$ be any straight line and F a fixed point. Let the points P, P', P'', . . . be so taken that each one is equally distant from the point F and from the line $D'D$. Then these points lie on a parabola of which the point F is the focus and the line $D'D$ is the directrix.

The line through the focus perpendicular to the directrix is called the **axis** of the parabola, and the point where the axis meets the curve is the **vertex**, V. A line through the focus terminated at both ends by the curve is called a **focal chord**; the focal chord at right angles to the axis, denoted by $L'L$ in the figure, is the **latus rectum.** A line drawn from the focus to a point on the parabola is called a **focal radius,** or simply a **radius.**

Note that, unlike the radii of a circle, the radii of a parabola differ greatly in length. The shortest radius is the one drawn to the vertex. As the radius is made to swing in either direction from this position it increases indefinitely in length until it coincides with the axis to the right, when it has no end-point except the focus.

56. Equation of the parabola.—Let $D'D$ be the directrix and F the focus of a parabola.



Place the origin of coördinates at the vertex of the parabola and make the x-axis fall along the axis of the parabola. Let $OF = p$. Since O is a point on the parabola, $OF = AO$. Hence $AO = p$.

Let $P(x, y)$ be any point on the parabola. Then, by definition,

$$FP = MP = p + x, \text{ or}$$

$$\sqrt{(x - p)^2 + y^2} = p + x.$$

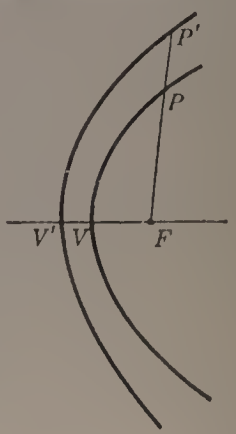
Squaring, cancelling, and transposing, this reduces to

$$y^2 = 4px \qquad [21]$$

which is the equation of the parabola in the standard form.

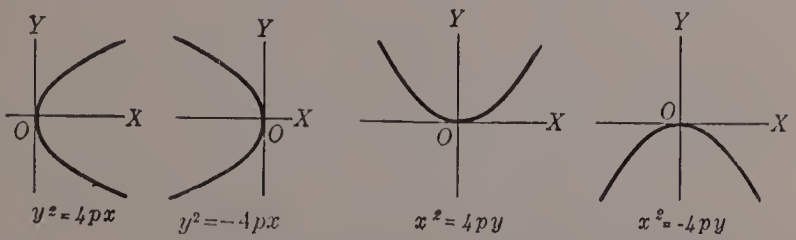
In this equation p is a positive number. Hence $4px$ is positive when x is positive, and negative when x is negative. But y is a real number only when $4px$ is positive and hence the curve represented by $y^2 = 4px$ cannot have any of its points to the left of the y-axis. As x increases indefinitely from zero, y increases indefinitely both in the positive and negative direction. Clearly the curve is symmetrical with respect to the x-axis.

57. General shape of parabola.—The distance from the focus to the directrix determines the scale on which the parabola is constructed, but the general shape of all parabolas is the same whatever is their scale.



Thus, if two parabolas are placed so that their axes and foci coincide and if V and V' are their vertices, then, using the notation shown in the figure, $FV : FV' = FP : FP'$. In this sense any two parabolas may be regarded as similar figures. See Example 3, p. 145

58. Special positions of the parabola.—If the equation of the parabola is put in the form $y^2 = -4px$, p being positive, the whole figure is turned about as shown in the figure. This change amounts simply to turning the figure around the y-axis through an angle of 180°.



If the equation is written in the form $x^2 = 4py$, the axis of the parabola coincides with the y-axis and the curve is open upward.

If the equation is written in the form $x^2 = -4py$, the curve is open downward.

Up to this point p in the equation $y^2 = 4px$ has been restricted to positive values. In general, however, p will be permitted to take any real values either positive or negative.

Thus, $y^2 = 4px$ may represent the parabola in its usual position or it may represent the second parabola in the figure above depending on the sign of p, while $x^2 = 4py$ may represent either of the last two parabolas in this figure.

59. Length of latus rectum.—To find the length of the latus rectum put $x = p$ in the equation $y^2 = 4px$.

Then $$y^2 = 4p^2 \text{ and } y = \pm 2p.$$

Hence the length of the latus rectum is $4p$.

SIGHT WORK

1. Find the equation in the form $y^2 = 4px$ of parabolas for each of the following values of p: 3, -4, $\sqrt{3}$, 8, $-\sqrt{5}$.

2. Find the latus rectum of each of the following parabolas: $y^2 = 6x$, $y^2 = 24x$, $y^2 = -16x$, $y^2 = -6kx$.

60. Tangent with given slope.—Clearly a tangent to the parabola meets it at only one point. Conversely, a line not parallel to the axis of the parabola which meets it in only one point is tangent to it. (Another definition of tangent is developed in Chapter X).

Let the equations of a parabola and of a tangent with a given slope m be,

$$y^2 = 4px \quad (1) \quad \text{and} \quad y = mx + b. \quad (2)$$

Substituting from (2) in (1), we have

$$m^2x^2 + x(2mb - 4p) + b^2 = 0.$$

Hence $$x = \frac{4p - 2mb \pm \sqrt{(2mb - 4p)^2 - 4m^2b^2}}{2m^2}.$$

If line (2) is to meet curve (1) in only one point, then

$$(2mb - 4p)^2 - 4m^2b^2 = 0, \text{ from which } b = p/m.$$

Substituting this value of b in (2), we have

$$y = mx + \frac{p}{m} \qquad [22]$$

This line is tangent to the parabola $y^2 = 4px$ for all values of m except $m = 0$. Compare tangent to circle, §50.

61. Tangent to parabola at a given point on it.— Let (x_1, y_1) be any point on the parabola $y^2 = 4px$. Then $y_1^2 = 4px_1$. Suppose $y = mx + p/m$ is tangent to the parabola at this point.

Then $$y_1 = mx_1 + p/m, \quad \text{or} \quad m^2x_1 - my_1 + p = 0.$$

Hence, $$m = \frac{y_1 \pm \sqrt{y_1^2 - 4px_1}}{2x_1} = \frac{y_1}{2x_1}.$$

Substituting $\frac{y_1^2}{4p}$ for its equal x_1 we have, $m = \frac{2p}{y_1}$.

Then the equation of the tangent is

$$y - y_1 = \frac{2p}{y_1}(x - x_1), \quad \text{or} \quad y_1y - y_1^2 = 2px - 2px_1.$$

Substituting $4px_1$ for its equal y_1^2 and transposing, we have

$$y_1y = 2px + 2px_1, \quad \text{or} \quad \boldsymbol{y_1y = 2p(x + x_1)} \qquad [23]$$

which is the equation of the tangent to the parabola $y^2 = 4px$ at the point (x_1, y_1). Compare tangent to circle §51.

EXERCISES

1. Find the value of p if the parabola $y^2 = 4px$ passes through the point (4, 12).

2. Find the value of p if the parabola $x^2 = 4py$ passes through the point (12, 4).

3. Find the values of p and p' if the parabolas $y^2 = 4px$ and $x^2 = 4p'y$ both pass through the point $(-8, 10)$. Sketch the parabolas.

4. Find the values of p and p' if the parabolas $y^2 = 4px$ and $x^2 = 4p'y$ both pass through the point $(-12, -16)$. Sketch the parabolas.

5. Find the equations of the tangents to the parabola $y^2 = 12x$ at the points whose abscissas are 3.

6. Find the equation of the tangent to the parabola $y^2 = 8x$ whose slope is $\frac{1}{2}$.

7. The parabola $y^2 = 4px$ passes through the point (6, 12); find the length of its latus rectum.

8. Find the value of p if the length of the latus rectum of the parabola $y^2 = -4px$ is 10.

9. A tangent to the parabola $y^2 = 10x$ is parallel to the line $3x - 5y = 6$; find its equation.

10. Find the equation of the tangent to the parabola $y^2 = 10x$ which is perpendicular to the line $3x - 5y = 6$.

11. Find the value of b if $y = mx + b$ is tangent to the parabola $x^2 = 4py$.

12. Find the equation of the tangent to $y^2 = 12x$ if its intercept on the y-axis is -4.

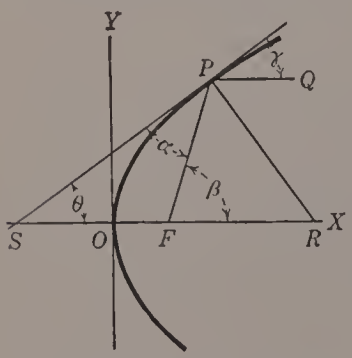
13. Find the equation of a tangent to $y^2 = -6x$ if its intercept on the x-axis is 8.

14. The length of the latus rectum of the parabola $y^2 = 4px$ is 8; find the equation of the tangent whose slope is $-3/5$.

15. Find the point on the parabola $y^2 = 16x$, the slope of the tangent at which is $3/4$.

16. Find the point on the parabola $y^2 = -8x$, the slope of the tangent at which is $-4/3$.

62. Normal to parabola.—The normal to a curve is a line perpendicular to the tangent at the point of tangency. The normal to a parabola at a point (x_1, y_1) is therefore found by finding the line at right angles to the tangent at that point.



The slope of the tangent at the point (x_1, y_1) is $\frac{2p}{y_1}$ (see §61).

Hence the slope of the normal PR at this point is $-\frac{y_1}{2p}$ (see §38), and the equation of the normal is

$$y - y_1 = -\frac{y_1}{2p}(x - x_1) \qquad [24]$$

63. Angle between tangent and focal chord through point of tangency.—Using the figure of §62, we wish to prove that angle α is equal to angle θ. The tangent at P, $y_1y = 2p(x + x_1)$, meets the x-axis in the point $S(-x_1, 0)$. Hence $SF = x_1 + p$. Also by the definition of the parabola $FP = x_1 + p$. Hence $\triangle SFP$ is isosceles and $\angle\theta = \angle\alpha$.

That is, the tangent to a parabola makes equal angles with the axis and with the focal radius to the point of tangency.

EXERCISES

1. Prove as follows that $\angle\theta = \angle\alpha$.

$$\beta = \alpha + \theta \text{ and hence } \alpha = \beta - \theta.$$

Then show (1) $\tan\beta = \frac{y_1}{x_1 - p}$; (2) $\tan\theta = \frac{2p}{y_1}$; (3) $\tan(\beta - \theta) = \frac{2p}{y_1}$.

See (S) §252.

2. Use the result obtained in §63 to construct a tangent to a parabola at any point on it when the focus and the axis of the parabola are given.

3. Construct the normal to a parabola at a given point on it when the axis and the focus are given.

64. An important geometric property of the parabola.—If the line PQ in the figure on page 78 is parallel to the axis, it follows that

$$\angle\gamma = \angle\theta = \angle\alpha.$$

Hence the normal PR makes equal angles with PF and PQ.

That is, a normal to a parabola bisects the angle formed by two lines drawn from the point of tangency, one to the focus and one parallel to the axis.

It is found in the study of optics that a beam of light is reflected from a curved surface precisely as it would be reflected from a plane tangent to the surface at the point of reflection. For this reason rays of light parallel to the axis meeting the parabola will all be reflected to the focus, and, conversely, a ray of light proceeding from the focus will be reflected along a line parallel to the axis. This property of the parabola is the foundation of the parabolic mirror used in headlights and in other optical instruments. (Compare §78.)

65. Tangent to parabola from point not on it; exterior and interior points.—The problem of finding the equation of a tangent to a parabola from a point not on it is illustrated in the following examples:

Example 1. Find the equations of the tangents from the point (6, 7) to the parabola $y^2 = 8x$.

SOLUTION: Using the equation $y = mx + p/m$ for the tangent with slope m we have, since $p = 2$,

$$y = mx + 2/m.$$

It is required that the tangents pass through the point (6, 7).

Hence $$7 = 6m + \frac{2}{m}, \quad \text{or} \quad 6m^2 - 7m + 2 = 0,$$

and $$m = \tfrac{2}{3} \text{ or } \tfrac{1}{2}.$$

Hence $y - 7 = \tfrac{2}{3}(x - 6)$ and $y - 7 = \tfrac{1}{2}(x - 6)$ are the required equations.

The problem may also be solved by solving simultaneously the equations $y^2 = 8x$ and $y - 7 = m(x - 6)$ and then finding the values of m which make the lines tangents.

Example 2. Find the equations of the tangents to the parabola $y^2 = 4x$ which pass through the point (6, 4).

SOLUTION: Using the formula $y = mx + \frac{p}{m}$ as in Example 1, substituting, and solving for m,

$$m = \frac{2 + \sqrt{-2}}{6}, \quad m = \frac{2 - \sqrt{-2}}{6}.$$

Since the slopes are imaginary there is no tangent to this parabola through the point (6, 4).

A point through which two tangents can be drawn to a parabola is an **exterior point,** and a point through which there is no tangent is an **interior point.**

EXERCISES

Group A

1. Find the equation of the form $y^2 = 4px$ of a parabola, the length of whose latus rectum is 24.

2. Find the equation of the tangent to the parabola $y^2 = 18x$ which is parallel to the line $2x - 5y + 3 = 0$.

3. Find the equation of the tangent to the parabola $y^2 = 14x$ which is perpendicular to the line $5x + 3y - 9 = 0$.

4. Find the equation of the tangent to $y^2 = 8x$ through (2, 4).

5. Find the equations of the tangents to $y^2 = 10x$ through (1, 5).

6. Find the equation of the normal to the parabola $y^2 = 6x$ which is perpendicular to the line $x - y + 9 = 0$.

7. Find the length of the chord which $y^2 = 10x$ cuts from $x + y = 6$.

8. Find the tangent to the parabola $y^2 = 12x$ which makes an angle of 30° with the axis.

9. Find the length of the chord which the parabola $y^2 = 4x$ cuts from the normal to it at the point (4, 4).

10. Find the distances from the focus and the vertex of the parabola $y^2 = 4px$ to the tangent at one end of the latus rectum; what is the ratio between these distances?

Group B

1. Find the equation of the form $y^2 = 4px$ of a parabola through the point (2, 6). What is the latus rectum of this parabola?

2. Find the equation of the tangent to the parabola $y^2 = 16x$ which is parallel to the line $2x - 7y - 9 = 0$.

3. Find the equation of the normal to the parabola $y^2 = 18x$ which is perpendicular to the line $x + y = 10$.

4. Find the equation of the tangent to the parabola $y^2 = 8x$ which passes through the point $(8, -8)$.

5. Find the equations of the tangents to the parabola $y^2 = 4x$ which pass through the point $(5, 6)$.

6. Find the length of the chord which $y^2 = 8x$ cuts from $x - y = 10$.

7. Find the points in which the two parabolas $y^2 = 4px$, $x^2 = 4py$ cut each other.

8. Find the normal to the parabola $y^2 = 6x$ which makes an angle of 150° with the positive direction of the axis.

9. Find the length of the chord which the parabola $y^2 = 8x$ cuts from the normal at the point $(2, 4)$.

10. The distance of a point P on the parabola $y^2 = 12x$ from the focus is 6 times the length of the latus rectum. Find the coördinates of P.

Group C

1. Find the equation of the form $y^2 = 4px$ of the parabola the length of whose latus rectum is 20.

2. Find the equation of the form $y^2 = 4px$ of the parabola which passes through the point $(6, 9)$.

3. Find the equation of the tangent to the parabola $y^2 = 10x$ which is parallel to the line $y - 2x + 7 = 0$.

4. Find the equation of the tangent to the parabola $y^2 = 12x$ which passes through the point $(3, -6)$.

5. Find the equations of the tangents to the parabola $y^2 = 10x$ which pass through the point $(-1, -1)$.

6. Find the length of the chord which the parabola $x^2 = -9y$ cuts from the line $x + y + 10 = 0$.

7. Find the equation of the normal to the parabola $y^2 = 20x$ if the normal makes an angle of 135° with the positive direction of the axis.

8. Show that the normal to a parabola at one end of a latus rectum is parallel to the tangent at the other end.

9. Find the diameter and center of the circle which passes through the vertex and the extremities of the latus rectum of the parabola $y^2 = 10x$.

10. What is the relation of p and p' if $y^2 = 4px$ and $x^2 = 4p'y$ cut each other at one extremity of the latus rectum of $y^2 = 4px$?

66. Solution of problems.—Methods of solving types of problems involving the parabola are illustrated in the following examples in which some further properties of the parabola are developed.

Example 1. Two points Q and R move on a parabola in such a way that their ordinates are in a constant ratio. Prove that the locus of the intersection of the tangents at Q and R is a parabola.

PROOF: If x_1, y_1 are the coördinates of Q then x_2, ky_1 are coördinates of R. Let the parabola be

$$y^2 = 4px. \tag{1}$$

The tangents to the parabola at $Q(x_1, y_1)$ and $R(x_2, ky_1)$ are

$$y_1y = 2p(x + x_1) \tag{2}$$

and

$$ky_1y = 2p(x + x_2). \tag{3}$$

Since (x_1, y_1) and (x_2, ky_1) are on (1),

$$x_1 = \frac{y_1^2}{4p} \quad (4) \qquad \text{and} \qquad x_2 = \frac{k^2y_1^2}{4p}. \tag{5}$$

At the point of intersection of (2) and (3),

$$y = \frac{2p}{y_1}(x + x_1) = \frac{2p}{ky_1}(x + x_2).$$

Substituting from (4) and (5),

$$\frac{2p}{y_1}\left(x + \frac{y_1^2}{4p}\right) = \frac{2p}{ky_1}\left(x + \frac{k^2y_1^2}{4p}\right). \tag{6}$$

Solving (6) for x, $\quad x = \dfrac{ky_1^2}{4p}$.

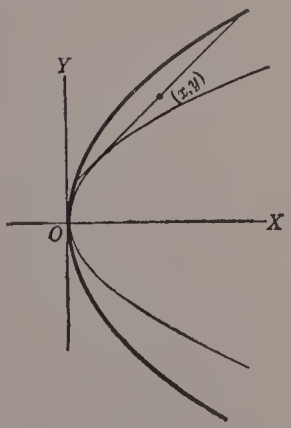
Substituting this value of x in (2) and using (4),

$$y = \frac{k+1}{2}y_1. \quad \text{Eliminating } y_1 \text{ from } x = \frac{ky_1^2}{4p} \text{ and } y = \frac{k+1}{2}y_1,$$

we have $y^2 = \dfrac{p(k+1)^2}{k}x$, which is the equation of a parabola.

Example 2. Two parabolas have the same axis and the same vertex. Prove that the middle points of chords of the larger parabola which are tangent to the smaller lie on a parabola.

PROOF: Let the parabolas be $y^2 = 4px$ (1) and $y^2 = 4apx$ (2), where $a > 1$. Let $y_1 y = 2p(x + x_1)$ (3), be any tangent to the smaller parabola, and let $(\bar{x}, \bar{y})$ be the middle point of the chord cut off on (3) by (2).



Eliminating y from (3) and (2),

$$x^2 + x(2 - 4a)\, x_1 + x_1^2 = 0.$$

Then, see §241,

$$\bar{x} = (2a - 1)x_1 = \frac{(2a - 1)y_1^2}{4p}.$$

Substituting this value of $\bar{x}$ for x in (3),

$$\bar{y} = ay_1.$$

Hence, suppressing dashes,

$$y^2 = \frac{4pa^2}{2a - 1}x,$$

which is the equation of a parabola.

Example 3. Pairs of tangents to a parabola make equal angles with a fixed line. Prove that the chords of contact of such pairs pass through a fixed point.

SOLUTION: Let $y^2 = 4px$ (1) be the parabola, and

let $y = mx$ (2) be the fixed line, or one parallel to it.

Lines making an angle with (2) whose tangent is k or $-k$ are

$$y = \frac{m + k}{1 - mk}x + \frac{p(1 - mk)}{m + k} \tag{3}$$

and

$$y = \frac{m - k}{1 + mk}x + \frac{p(1 + mk)}{m - k}. \tag{4}$$

The point of tangency of (3) is $\left(\frac{p(1 - mk)^2}{(m + k)^2}, \frac{2p(1 - mk)}{m + k}\right)$;

and that of (4), $\left(\frac{p(1 + mk)^2}{(m - k)^2}, \frac{2p(1 + mk)}{m - k}\right)$.

The line joining these two points is

$$\frac{(m - k)y - 2p(1 + mk)}{(m - k)^2 x - p(1 + mk)^2} = \frac{(m + k)}{m(1 + k^2)}. \tag{5}$$

Now assume two values for k; e.g., $k = 0$ and $k = -\frac{1}{m}$. Find the intersection point of these two lines, and show that it lies on the line (5).

MISCELLANEOUS PROBLEMS ON CHAPTER V

1. Find the locus of the centers of circles which are tangent to a fixed circle and to a fixed line.

SUGGESTION: Construct a line l parallel to the given line and at a distance from it equal to the radius of the given circle. Make l and the center of the circle lie on opposite sides of the given line.

Consider the locus of a point P which is equidistant from l and the center of the given circle.

2. Find the locus of the centers of circles which are tangent to a fixed line and which pass through a fixed point.

3. Prove that the ordinate of the point of intersection of two tangents to a parabola is equal to half the sum of the ordinates of the points of contact of the tangents, and that the abscissa is the geometric mean of the abscissas of the points of contact.

4. Prove that the slope of the line joining the vertex of the parabola $y^2 = 4px$ to the point of intersection of any two tangents is the sum of the slopes of the tangents.

5. Prove that a tangent to a parabola meets the directrix and the latus rectum (extended) in points which are equidistant from the focus.

6. Show that any circle described on a focal chord as diameter is tangent to the directrix of the parabola.

7. Show that the circle constructed on any focal radius as a diameter is tangent to the line which is tangent to the parabola at the vertex.

8. Prove that if r is the distance from the focus to the point P on a parabola $y^2 = 4px$ and if a is the perpendicular distance from the focus to the tangent at P, then $a^2 = pr$.

9. Prove that if tangents are drawn to the parabola $y^2 = 4px$ from a point on the line $x = -4p$, their chord of contact will subtend a right angle at the vertex.

10. Find the equation of the locus of the points in which perpendiculars from the focus to the normals to a parabola meet the normals.

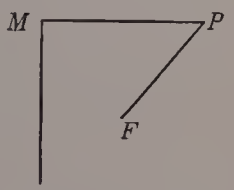
11. The chord PQ is normal to the parabola at P. The tangents at P and Q meet in T. Show that PT is bisected by the directrix.

12. The line joining the focus to the point of intersection of two tangents bisects the angle formed by the lines joining the focus to the points of tangency.

CHAPTER VI

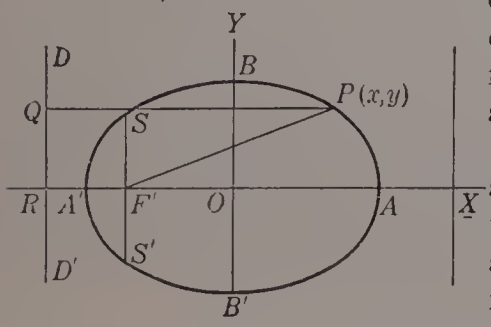
THE ELLIPSE

The **ellipse** is the locus of all points whose distances from a fixed point and from a fixed line are in a constant ratio which is less than unity. In this definition "distance" does not involve direction; in it all distances are regarded as simply "length." As in the case of the parabola, the fixed line is called the **directrix,** and the fixed point the **focus.** The constant ratio is denoted by e. In the case of the ellipse $0 < e < 1$.

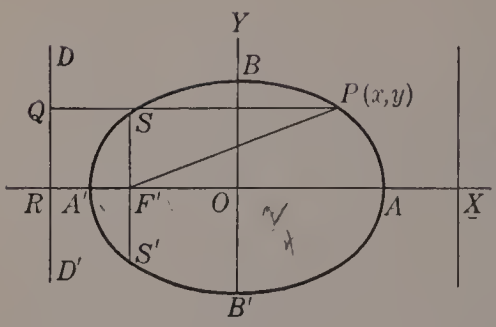


If P is a point on the ellipse, F the focus, and MP the perpendicular distance from the directrix, then $FP : MP = e$. The number e is called the **eccentricity** of the ellipse.

67. Choice of axes.—The equation of the ellipse and the work of deriving it may be simplified by a certain choice of axes.



Let $D'D$ be the directrix and F' the focus. Through the focus draw a line at right angles to the directrix meeting it in R. Then there is a point A' between R and F' such that $F'A' : RA' = e$. Also there is a point A to the right of F' for which $F'A : RA = e$. The points A' and A are the **vertices** of the ellipse. Let O be the middle point of the segment $A'A$. This point, which is the **center** of the ellipse, will be taken as the origin, and the line $A'A$ as the x-axis.



68. Equation of the ellipse.—In the figure $D'D$ is the directrix, F' is the focus, and A' and A are points on the ellipse. The origin O is the middle point of $A'A$.

Let $A'O = a$. Then the x-coördinates of A' and A are $-a$ and a respectively.

By the definition of the ellipse,

$$\frac{F'A'}{RA'} = e \quad \text{and} \quad \frac{F'A}{RA} = e.$$

But $F'A' = a - F'O$, $RA' = RO - a$, $F'A = F'O + a$, $RA = RO + a$.

Then, $$\frac{a - F'O}{RO - a} = e, \quad \text{or} \quad a - F'O = eRO - ae. \tag{1}$$

Also $$\frac{F'O + a}{RO + a} = e, \quad \text{or} \quad a + F'O = eRO + ae. \tag{2}$$

Adding (1) and (2), $2a = 2eRO$, or $RO = \frac{a}{e}$.

Subtracting (1) from (2), $2F'O = 2ae$, or $F'O = ae$.

Hence R is the point $\left(-\frac{a}{e}, 0\right)$, F' is $(-ae, 0)$, and $x = -\frac{a}{e}$ is the equation of the directrix.

Let $P(x, y)$ be any point on the ellipse.

Then $F'P = \sqrt{(x + ae)^2 + y^2}$ and $QP = x + \frac{a}{e}$.

But by definition, $\frac{F'P}{QP} = e$ or $F'P = eQP$.

Hence $\sqrt{(x + ae)^2 + y^2} = e\left(x + \frac{a}{e}\right) = ex + a$.

Squaring and reducing,

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2,$$

or $$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

Dividing by $a^2(1 - e^2)$, $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

Since $e < 1$, $1 - e^2$ is positive and hence $a^2(1 - e^2)$ is positive. The quantity $a^2(1 - e^2)$ is denoted by b^2.

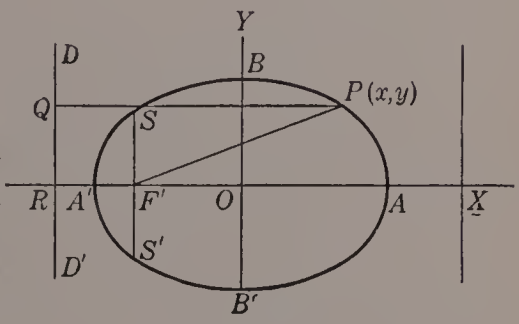
Then $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad [24]$$

This is the standard form of the equation of the ellipse.

69. Simple properties of the ellipse.—If in the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$x = 0$, then $y = \pm b$. Hence the ellipse cuts the y-axis in two points each at a distance b from the origin. We have already seen that it cuts the x-axis in two points each at a distance a from the origin. The segments $A'A$ and $B'B$ in the figure are called respectively the **major** and **minor axes** of the ellipse, OA and OB being the **semimajor** and **semiminor axes**. Hence a and b are the lengths of the semiaxes.



It is evident from the equation that for $x = 0$, y has its greatest numerical values, and that as x increases from 0 to a or decreases from 0 to $-a$, the absolute value of y decreases until it reaches zero for $x = a$ and for $x = -a$.

Since x and y occur in the equation as squares only, the curve is symmetrical with respect to both axes and hence it follows that it is symmetrical with respect to the origin.

70. Directrices, foci, latus rectum.—It is an immediate consequence of the symmetry of the curve with respect to the y-axis that the line $x = a/e$ and the point $(ae, 0)$ may also be used as the directrix and the focus. Hence the ellipse has two directrices, $x = \pm a/e$, and two foci, $(\pm ae, 0)$.

The double ordinate $S'S$ through one of the foci is called the **latus rectum.** Its length is found by putting $x = ae$ in the equation of the ellipse.

Then, $y^2 = b^2(1 - e^2) = \dfrac{a^2b^2(1 - e^2)}{a^2} = \dfrac{b^4}{a^2}$, and $y = \dfrac{b^2}{a}$.

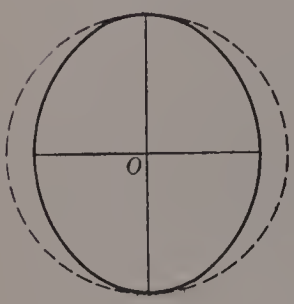
Hence $S'S = \dfrac{2b^2}{a}$, which is the length of the latus rectum.

The facts stated in this section are frequently used in dealing with the ellipse. They are:

$$\textbf{Foci: } (\pm ae, 0), \text{ or } (\pm\sqrt{a^2 - b^2}, 0) \qquad [26]$$

$$\textbf{Directrices: } x = \pm\frac{a}{e} \qquad [27]$$

$$\textbf{Length of latus rectum is } \frac{2b^2}{a} \qquad [28]$$



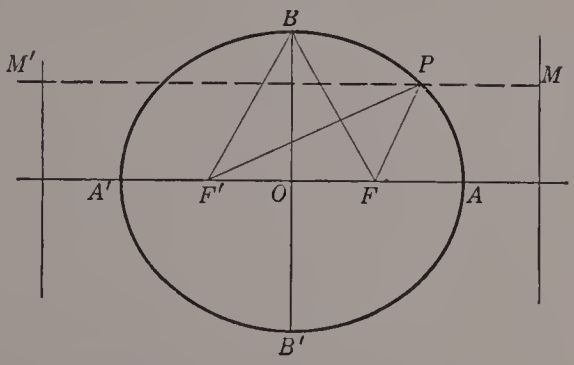
71. Major axis on the y-axis.—If in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

b is greater than a, then the ellipse has its major axis on the y-axis as shown in the figure. In this case a is the length of the semiminor axis and b is the length of the semimajor axis, (see § 73). The foci are then on the y-axis at a distance $\sqrt{b^2 - a^2}$ from the origin. The latus rectum is the double abscissa through a focus and its length is $\dfrac{2a^2}{b}$.

If $a = b$ the equation reduces to $x^2 + y^2 = a^2$ and the locus is a circle.

72. A fundamental property of the ellipse.—Let $P(x, y)$ be any point on the ellipse and let F' and F be its foci. Then it can be shown that the sum of $F'P$ and FP is the same for all such points P.



Let M' and M be on the directrices. Then by the definition of the ellipse

$$F'P = eM'P = e\left(\frac{a}{e} + x\right) = a + ex,$$

and

$$FP = ePM = e\left(\frac{a}{e} - x\right) = a - ex.$$

Hence,

$$F'P + FP = 2a.$$

The converse of this result, namely that if $F'P + FP = 2a$, then the point P is on the ellipse is easily proved.

This is one of the important properties of the ellipse and is often used as the basis for its definition.

That is, *an ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is a constant.*

It follows that $F'B = a$ and hence that the distance from the center to the focus is $\sqrt{a^2 - b^2}$.

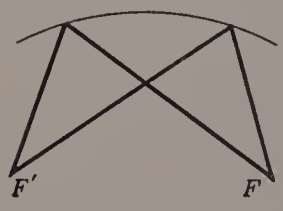
73. Eccentricity in terms of a and b.—In the figure of §72,

$$F'O = ae,\ OB = b,\ F'B = a.$$

Hence, $a^2e^2 + b^2 = a^2$ and $e = \frac{1}{a}\sqrt{a^2 - b^2}$ [29]

In case b is greater than a, then $a^2 = b^2(1 - e^2)$ and hence the eccentricity is $\frac{1}{b}\sqrt{b^2 - a^2}$.

74. Mechanical construction of an ellipse.—When the foci and the major axis are given, the ellipse may be constructed as follows: Fasten pins in the paper at the foci. Tie a string to them, making the length of the string between the pins equal to $2a$, the length of the major axis. Then press a pencil against the string and move it, keeping the string taut. The sum of the distances of the pencil point from the two pins will at each point be equal to the length of the string; hence by §72, the figure traced will be an ellipse.



This method is sometimes called the gardener's method for constructing the ellipse since it is used to trace the outline of elliptical flower beds.

EXERCISES

1. Find the semiaxes, the latus rectum, the foci, and vertices of the ellipse $4x^2 + 9y^2 = 36$. Sketch the curve.

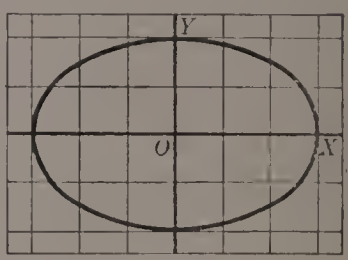
It is evident that the semiaxes are 3 and 2.

Then $\frac{2b^2}{a} = \frac{8}{3} = 2\frac{2}{3}$ is the length of the latus rectum.

The foci, $(\pm\sqrt{a^2 - b^2}, 0)$, are $(\pm\sqrt{5}, 0)$.

The following points of the curve are known at once:

Vertices $(\pm 3, 0)$; ends of minor axis $(0, \pm 2)$; ends of latus rectum at right $\left(\sqrt{5}, \pm\frac{4}{3}\right)$, and at the left $\left(-\sqrt{5}, \pm\frac{4}{3}\right)$.



Hence the curve is readily sketched.

Find the semiaxes, latus rectum, eccentricity, foci, and vertices of each of the ellipses in Exs. 2 to 7.

2. $16x^2 + 25y^2 = 400$

3. $x^2 + 9y^2 = 9$

4. $4x^2 + 49y^2 = 196$

5. $20x^2 + 10y^2 = 30$

6. $16x^2 + 12y^2 = 3$

7. $x^2 + 5y^2 = 12$

Give in the standard form the equation of the ellipse for which

8. $a = 5$; $b = 3$

9. $a = 5$; $e = 3/5$

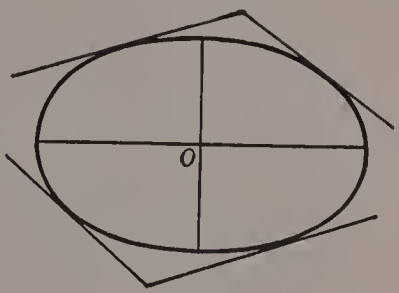
10. $b = 12$; $e = 5/13$

11. $a = 9$; latus rectum $= 8$

12. $b = 7$; foci $(\pm 24, 0)$

13. vertices $(\pm\sqrt{17}, 0)$; foci $(\pm 2, 0)$

75. Tangent with given slope.—It is evident from the form of the ellipse that there are two tangents to it parallel to any given line. That is, for any fixed value of m there are two values of c for which the line $y = mx + c$ is a tangent. The problem is to find these two values of c.



(Notice that $y = mx + c$ is used instead of the usual notation, $y = mx + b$. This is done to avoid confusion since b is used in the equation of the ellipse to represent a different quantity.)

Eliminating y in the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad y = mx + c,$$

we have $x^2(b^2 + m^2a^2) + 2a^2cmx + a^2(c^2 - b^2) = 0.$

To make the line tangent to the ellipse the left member of this equation must be a square.

Hence, $(2a^2cm)^2 - 4(b^2 + m^2a^2)a^2(c^2 - b^2) = 0,$

or $4a^4c^2m^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4b^2m^2 = 0.$

(See §242)

Cancelling and solving for c, we have

$$c = \pm\sqrt{a^2m^2 + b^2}.$$

Hence the equations of the tangents with slope m are

$$y = mx + \sqrt{a^2m^2 + b^2}, \quad y = mx - \sqrt{a^2m^2 + b^2} \qquad [30]$$

SIGHT WORK

Find the equations of the tangents to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$:

(a) when the slopes of the tangents are $\frac{2}{3}$;
(b) when the tangents are parallel to the line $x - 3y + 6 = 0$;
(c) when the tangents are perpendicular to the line $2x + y - 9 = 0$.

76. Tangent at a given point on the ellipse.—The problem of finding the equation of the tangent to the ellipse at a given point on it is solved in the same manner as for the parabola.

By §75, $y = mx + \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse for all values of m. If (x_1, y_1) is the point of tangency, then

$$y_1 = mx_1 + \sqrt{a^2m^2 + b^2}, \tag{1}$$

and it remains to find the value of m.

$$y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2m^2 + b^2,$$

or
$$m^2(a^2 - x_1^2) + 2x_1y_1m + b^2 - y_1^2 = 0.$$

Hence
$$m = \frac{-2x_1y_1 \pm \sqrt{4x_1^2y_1^2 - 4(a^2 - x_1^2)(b^2 - y_1^2)}}{2(a^2 - x_1^2)},$$

which reduces to

$$m = \frac{-x_1y_1 \pm \sqrt{b^2x_1^2 + a^2y_1^2 - a^2b^2}}{a^2 - x_1^2}.$$

If (x_1, y_1) is on the ellipse, then $b^2x_1^2 + a^2y_1^2 = a^2b^2$, and this makes the expression under the radical equal to zero.

Hence,
$$m = -\frac{x_1y_1}{a^2 - x_1^2}. \tag{2}$$

But from
$$b^2x_1^2 + a^2y_1^2 = a^2b^2,$$

$$a^2 - x_1^2 = \frac{a^2y_1^2}{b^2}.$$

Substituting this value of $a^2 - x_1^2$ in (2) we have

$$m = -\frac{b^2x_1}{a^2y_1} \tag{31}$$

Hence the tangent is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1),$$

which reduces to

$$b^2x_1x + a^2y_1y = b^2x_1^2 + a^2y_1^2 = a^2b^2, \quad \text{or} \quad \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \tag{32}$$

Compare this equation with the equation of the tangent to the circle given in §51.

EXERCISES

(Unless otherwise stated the equations of the ellipses in these exercises are in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $b^2 < a^2$ (with major axis on the x-axis).

Group A

1. In the equation of an ellipse $a = 7$, $b = 4$; find the equations of the tangents whose slope is $\frac{2}{3}$.

2. In the ellipse $9x^2 + 25y^2 = 225$ find the equations of tangents parallel to the line joining the positive extremities of the axes.

3. In the ellipse of Ex. 2 find the equation of the tangent at the upper end of the ordinate for which $x = 4$.

4. An ellipse passes through the point (4, 3); find the value of a if $b = 4$.

5. An ellipse passes through the point (8, 3); find the value of b if $a = 10$.

6. Find the equation of the ellipse which passes through the points (7, 5), (9, − 3).

SUGGESTION: $\frac{49}{a^2} + \frac{25}{b^2} = 1$, $\frac{81}{a^2} + \frac{9}{b^2} = 1$; solve for $\frac{1}{a^2}$ and $\frac{1}{b^2}$.

7. The sum of the focal distances from a point P on an ellipse is 16. If $b = 6$, find the equation of the ellipse.

8. Find the equations of the tangents to the ellipse $\frac{x^2}{49} + \frac{y^2}{25} = 1$ at the points for which $y = 4$.

9. Find the equation of an ellipse whose latus rectum is 8 if the ellipse passes through the point (0, 6).

10. Find the equation of an ellipse whose latus rectum is 8 if the sum of the focal distances of a point on it is 14.

11. In an ellipse whose foci are (− 4, 0), (4, 0) the sum of the focal distances of a point on it is 12; find the equation of the ellipse.

Group B

1. If in an ellipse $a = 8$, $b = 6$, find the equations of the tangents whose slope is $-\frac{3}{4}$.

2. Find the latus rectum and the eccentricity of the ellipse in Ex. 1.

3. Find the equation of the ellipse which passes through the points (3, 1), (1, 2).

4. In an ellipse $b = 5$ and the sum of the focal radii is 12. Find the equations of the tangents through the points with ordinate 3.

5. Find the equation of an ellipse of which (4, 0) is a focus and $x = 9$ is the corresponding directrix.

6. Find the equation of an ellipse whose minor axis is 12 and whose latus rectum is 9.

7. Find the equation of an ellipse whose minor axis is 10 and one of whose foci is (5, 0).

8. Find the equations of the tangents to the ellipse $16x^2 + 25y^2 = 400$ which are parallel to the line joining the positive end of the major axis and the negative end of the minor axis.

9. Find the equation of the tangent to the ellipse $49x^2 + 16y^2 = 784$ at the lower end of the ordinate for which $x = 2$.

10. Find the equation of an ellipse with eccentricity 3/5 which passes through the point (5, 3).

11. Find the equation of an ellipse in which the sum of the focal distances of one point is 14 and one of whose foci is (5, 0).

Group C

1. If an ellipse $a = 8$, $b = 4$, find the equations of the tangents at the points whose ordinates are 3.

2. Find the equations of the tangents to the ellipse of Ex. 1 which are parallel to the line drawn through the negative ends of the axes.

3. Find the equations of the tangents to the ellipse in Ex. 1 which have a slope 4/5.

4. Find the equation of the ellipse which passes through the points (4, 2), (− 2, 3).

5. Find the equation of the ellipse with eccentricity $\frac{5}{6}$ which passes through the point (7, − 3).

6. Find the equation of the ellipse with eccentricity $\frac{3}{4}$ of which $x = -8$ is a directrix.

7. An ellipse has a focus (8, 0) and passes through the point (− 12, 0); find its equation.

8. An ellipse has $\frac{3}{4}$ as its eccentricity, and (− 6, 0) as a focus; find its equation.

9. The length of the latus rectum of an ellipse is 18 and the sum of the focal radii is 32; find its equation.

10. The sum of the focal radii to a point on an ellipse is 20, and (7, 0) is one of the foci; find the equation of the ellipse.

11. The line $x = 10$ is the directrix of an ellipse and (6, 0) is a focus; find the equation of the ellipse.

77. Angles formed by the focal radii with the tangent and the normal.—The slope of the normal to the ellipse (§76) at the point (x_1, y_1) is $\frac{a^2y_1}{b^2x_1}$, and the slopes of the focal radii are $\frac{y_1}{x_1 + ae}$ and $\frac{y_1}{x_1 - ae}$.

Hence the tangent of the angle between the normal and one of the focal radii is (see (10), §252).

$$\frac{\frac{a^2y_1}{b^2x_1} - \frac{y_1}{x_1 + ae}}{1 + \frac{a^2y_1^2}{b^2x_1(x_1 + ae)}} = \frac{(a^2 - b^2)x_1y_1 + a^3ey_1}{a^2y_1^2 + b^2x_1^2 + b^2x_1ae}$$

$$= \frac{a^2e^2x_1y_1 + a^3ey_1}{a^2b^2 + b^2x_1ae} = \frac{aey_1(a^2 + x_1ae)}{b^2(a^2 + x_1ae)} = \frac{aey_1}{b^2}.$$

In the same manner it can be shown that the tangent of the angle formed by the normal and the other focal radius is $\frac{aey_1}{b^2}$.

Hence these angles are equal, and we have:

The normal to an ellipse at a point makes equal angles with the focal radii to this point.

Since the tangent and the normal form a right angle we have:

The tangent to an ellipse makes equal angles with the focal radii drawn to the point of contact.

These properties of the tangent and normal may also be stated as follows:

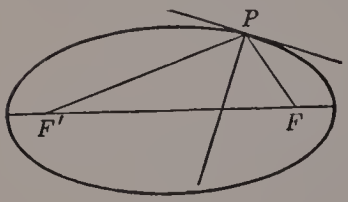
The tangent and the normal to an ellipse at a point are the bisectors of the angles formed by the focal radii drawn to this point.

The student may show that the tangent of the angle formed by the tangent and a focal radius is $\frac{b^2}{aey_1}$.

EXERCISE

If the foci of an ellipse and a point on it are given, show how to construct a normal and a tangent to the ellipse at this point.

78. Optical properties of foci.—In §77 we found that the tangent to an ellipse (and also the normal) makes equal angles with the focal radii drawn to the point of contact. Using the principle of optics stated in §64, it follows that a beam of light from one focus F meeting the ellipse at a point P will be reflected to the other focus F'.



79. Comparison of ellipse, parabola, and circle.—In §64 it was found that a beam of light from the focus of a parabola is reflected by the parabola along a line parallel to the axis. Hence a parabola may be regarded as the limiting curve of the ellipse when one of its foci is made to recede indefinitely far along its axis. On the other hand, a radius of a circle is perpendicular to the tangent at the point of contact and hence a beam of light from the center of the circle is reflected back to the center. The circle may therefore be regarded as the limiting curve of the ellipse when the two foci are made to approach the center as a limit.

This may be seen from the equation of the ellipse which may be written in the form

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2) \quad \text{(see §68)}.$$

If e is made to approach zero the coördinates of the foci $(\pm ae, 0)$ will approach $(0, 0)$. That is, the foci will approach the center. Also $1 - e^2$ approaches 1 and the equation approaches $x^2 + y^2 = a^2$. Hence the eccentricity of the circle is zero.

If e is made to approach unity the equation approaches $y^2 = 0$ which represents the x-axis. To make the ellipse approach the parabola it is necessary not only to make e approach zero, but both a and b must be made to increase indefinitely in a certain way. This is considered further in Chapter XV, see page 223.

EXERCISES

Unless otherwise stated the equations of the ellipses in these exercises are of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $b^2 < a^2$.

Group A

1. Find the eccentricity, the coördinates of the foci, and the equations of the directrices of the ellipse $\frac{x^2}{49} + \frac{y^2}{16} = 1$.

2. Verify that the sum of the distances from the point $\left(4, 2\frac{2}{5}\right)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ to its foci is equal to 10.

3. Find the equations of those tangents to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ which are parallel to the line $2x - 7y = 4$.

4. Find the points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, the tangents at which are perpendicular to the line $2x + y - 9 = 0$. SUGGESTION: Use [31]

5. In the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$ the tangent at the point $(4, \sqrt{5})$ and one of the focal radii drawn to this point form an angle, α; find the tangent of α.

6. In the ellipse of Ex. 5 find the angle formed by a focal radius and a tangent drawn to a point whose abscissa is -5.

7. Decide whether the point (5, 3) lies inside or outside the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

8. Find the equation of the ellipse whose eccentricity is $\frac{2}{3}$ and whose foci are $(\pm 4, 0)$.

9. Find the equation of the ellipse which passes through the points (6, 3) and (4, 4).

10. Find the equation of the ellipse whose semimajor axis is 5 and whose latus rectum is 5.

11. The eccentricity of an ellipse is $\frac{5}{7}$ and its foci are the points $(\pm 5, 0)$; find its equation.

12. Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the tangents at which are parallel to the line $\frac{x}{a} + \frac{y}{b} = 1$.

Group B

1. Find the equations of the tangents to the ellipse $\frac{x^2}{14} + \frac{y^2}{18} = 1$ which are perpendicular to the line $x - y + 2 = 0$.

2. Find the points at which tangents parallel to the line $2x - y + 9 = 0$ meet the ellipse $\frac{x^2}{19} + \frac{y^2}{24} = 1$. SUGGESTION: Use [31].

3. Find the eccentricity, coördinates of the foci, and equations of the directrices of the ellipse $\frac{x^2}{30} + \frac{y^2}{15} = 1$.

4. Find the tangent of the angle formed by the focal radii of the ellipse $\frac{x^2}{20} + \frac{y^2}{8} = 1$ to a point whose ordinate is 2.

5. Find the equations of the tangents to the ellipse in Ex. 4 which are parallel to the line through the positive ends of its two axes.

6. Find the equation of the ellipse whose minor axis is 8 and whose latus rectum is 6.

7. Find the equation of the ellipse which passes through the points (6, 3), (2, 6).

8. Find the equation of an ellipse whose minor axis is 10 and whose eccentricity is $\frac{2}{5}$.

9. The major axis of an ellipse is 12 and the latus rectum is 4; find the equation of the ellipse.

10. Decide which of the points (3, 9), (2, 4), (− 6, 2), (3, 2), (− 6, 5) are inside and which are outside the ellipse $\frac{x^2}{60} + \frac{y^2}{40} = 1$.

11. Find the points on the ellipse $\frac{x^2}{30} + \frac{y^2}{12} = 1$ the tangents at which are parallel to the line $\frac{x}{\sqrt{30}} + \frac{y}{\sqrt{12}} = 1$.

12. Prove that the point (x_1, y_1) is inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1$ and outside the ellipse if $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1$.

13. Verify that the sum of the distances from the foci to the point $\left(\frac{5\sqrt{3}}{3}, 2\right)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{6} = 1$ is equal to the major axis.

Group C

1. Find the eccentricity, the coördinates of the foci, and the equations of the directrices of the ellipse $\frac{x^2}{150} + \frac{y^2}{81} = 1$.

2. Find the equations of the tangents to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ which are perpendicular to the line $\frac{x}{4} + \frac{y}{3} = 1$.

3. Find the points of contact of the tangents to the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ which are parallel to the line $\frac{x}{6} + \frac{y}{4} = 1$.

4. Find the equation of the ellipse whose foci are $(\pm 5, 0)$ and whose latus rectum is 8.

5. Find the equation of the ellipse whose major axis is 12 and which passes through the point (5, 2).

6. Find the equation of the ellipse whose minor axis is 10 and whose directrices are $x = \pm 12$.

7. Find the equation of the ellipse which passes through the points (5, 6) and (7, 3).

8. Find the points on the ellipse $\frac{x^2}{40} + \frac{y^2}{16} = 1$ at which the tangents have the slope $-\frac{3}{5}$. SUGGESTION: Use [31].

9. Find the equation of the ellipse whose foci are $(\pm 8, 0)$ and whose eccentricity is $\frac{3}{5}$.

10. Verify that the sum of the distances from the foci of the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$ to the point $\left(2, \frac{4}{3}\sqrt{10}\right)$ is equal to the major axis.

11. The eccentricity of an ellipse is $\frac{2}{7}$ and its directrices are $x = \pm 9$; find its equation.

12. Find the points on the ellipse $\frac{x^2}{64} + \frac{y^2}{25} = 1$, the normal at each of which has a slope $\frac{2}{3}$.

80. Solution of problems.—Methods of solving types of problems involving the ellipse are illustrated in the following examples in which some further properties of the ellipse are developed.

Example 1. Two tangents to an ellipse are perpendicular to each other; find the locus of their point of intersection.

SOLUTION: Let m and $-\frac{1}{m}$ be the slopes of the tangents.

Then their equations are (§75)

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \text{and} \quad y = -\frac{x}{m} + \sqrt{\frac{a^2}{m^2} + b^2}.$$

These reduce to

$$y^2 - 2mxy + m^2x^2 = a^2m^2 + b^2 \tag{1}$$

and

$$y^2 + \frac{2xy}{m} + \frac{x^2}{m^2} = \frac{a^2}{m^2} + b^2,$$

or

$$m^2y^2 + 2mxy + x^2 = a^2 + m^2b^2. \tag{2}$$

Adding (1) and (2) we have

$$y^2(1 + m^2) + x^2(1 + m^2) = (a^2 + b^2)(1 + m^2),$$

and hence,

$$x^2 + y^2 = a^2 + b^2.$$

That is, the locus is a circle whose radius is $\sqrt{a^2 + b^2}$ and whose center is at the center of the ellipse.

Example 2. Prove that the product of the perpendiculars from the foci upon a tangent to the ellipse is equal to the square of the semiminor axis.

PROOF: The equation of the tangent at the point $P(x_1, y_1)$ is

$$b^2x_1x + a^2y_1y - a^2b^2 = 0. \tag{1}$$

The foci are $F'(-ae, 0)$, $F(ae, 0)$ and the distances from the foci to the tangents are

$$-\frac{b^2x_1ae + a^2b^2}{\sqrt{b^4x_1^2 + a^4y_1^2}} \quad \text{and} \quad \frac{b^2x_1ae - a^2b^2}{\sqrt{b^4x_1^2 + a^4y_1^2}}$$

and their product is

$$-\frac{b^4x_1^2a^2e^2 - a^4b^4}{b^4x_1^2 + a^4y_1^2}.$$

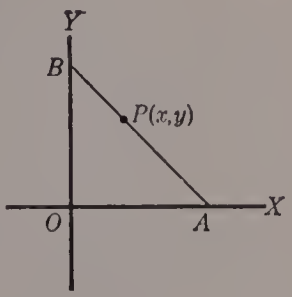
Substituting

$$e^2 = \frac{a^2 - b^2}{a^2} \quad \text{and} \quad y_1^2 = \frac{b^2}{a^2}(a^2 - x_1^2),$$

this reduces to

$$-\frac{b^2(b^2x_1^2a^2 - b^4x_1^2 - a^4b^2)}{b^4x_1^2 + a^4b^2 - a^2b^2x_1^2} = b^2.$$

Example 3. A line of fixed length moves so that its ends are on the coördinate axes. Find the locus of a point P on this line which divides it in a fixed ratio. Is this locus a circle in any case?



SOLUTION: Let the points A and B be $(x_1, 0)$, $(0, y_1)$. Denote the length of the segment AB by p and let $P(x, y)$ divide it so that $AP : PB = m : n$.

Then $\dfrac{n}{m+n} = \dfrac{x}{x_1}$ and $\dfrac{m}{m+n} = \dfrac{y}{y_1}$.

Hence $x_1 = \dfrac{(m+n)x}{n}$ and $y_1 = \dfrac{(m+n)y}{m}$.

But $x_1^2 + y_1^2 = p^2$.

Hence $$\frac{(m+n)^2x^2}{n^2} + \frac{(m+n)^2y^2}{m^2} = p^2,$$

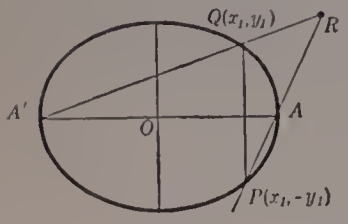
and the locus is an ellipse whose semiaxes are $\dfrac{np}{m+n}$ and $\dfrac{mp}{m+n}$.

If $m = n$ this equation reduces to $x^2 + y^2 = \dfrac{m^2p^2}{4m^2} = \dfrac{p^2}{4}$.

Example 4. A chord PQ is perpendicular to the major axis of an ellipse whose vertices are A', A. Find the locus of the intersection point R of $A'Q$ and AP.

If Q is the point (x_1, y_1) then P is $(x_1, -y_1)$ and the equations of the lines $A'Q$ and AP are

$$A'Q: \frac{y - y_1}{x - x_1} = \frac{y_1}{x_1 + a}, \qquad AP: \frac{y + y_1}{x - x_1} = \frac{-y_1}{x_1 - a},$$



or $$A'Q: y_1(a + x) = y(a + x_1),$$
$$AP: y_1(a - x) = y(x_1 - a).$$

Solving for x_1 and y_1, $x_1 = \dfrac{a^2}{x}$, $y_1 = \dfrac{ay}{x}$.

Substituting in $b^2x_1^2 + a^2y_1^2 = a^2b^2$, we have,

$$b^2a^2 + a^2y^2 = b^2x^2,$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

MISCELLANEOUS PROBLEMS ON CHAPTER VI

1. Find the equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of the ellipse whose major axis is 12 if the foci are midway between the center and the vertices.

2. Prove that if the normal to an ellipse, not meeting it on one of the axes, passes through the center, then $a = b$ and the curve is a circle.

3. Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the tangent at each point makes equal angles with the two axes.

4. An ellipse moves so that it is always tangent to the two coördinate axes; prove that the locus of its center lies on a circle.

5. The points A', A are the vertices of an ellipse. The tangent at a point P meets the tangent at A in Q. Show that OQ is parallel to PA', O being the center of the ellipse.

6. Prove that the perpendicular from a focus to a tangent meets the line joining the center to the point of contact on one of the directrices.

7. Prove that if a point P moves along an ellipse, then the distance from P to the center decreases constantly as P moves from an extremity of the major axis to an extremity of the minor axis.

8. If two ellipses have the same vertices show that for any fixed value of x, the values of y are in the same ratio as their minor axes.

9. Find the locus of the middle points of all chords drawn through the positive end of the minor axis of an ellipse.

10. $P'P$ is a double ordinate of an ellipse whose center is O. The normal at P' meets OP in Q. Show that the locus of Q is an ellipse.

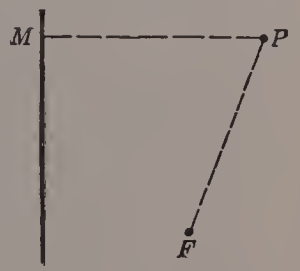
11. If the normal to an ellipse at P meets the major axis in Q, show that the locus of the middle point of PQ is an ellipse.

12. P is any point on an ellipse whose vertices are A', A. PN is drawn perpendicular to AP and PM perpendicular to $A'P$, the points M and N being on the x-axis. Show that MN is equal to the latus rectum.

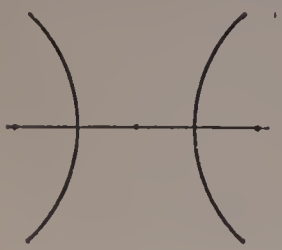
CHAPTER VII

THE HYPERBOLA

The **hyperbola** is the locus of all points whose distances from a fixed point and from a fixed line are in a constant ratio which is greater than unity. This fixed ratio, called the **eccentricity,** is denoted by e. As in the cases of the parabola and the ellipse, the fixed point is called the **focus** and the fixed line the **directrix.** If P is a point on the hyperbola, F the focus, and MP the distance from P to the directrix, then $FP : MP = e$. We shall see presently that a hyperbola resembles the ellipse in that it has two foci and two directrices.



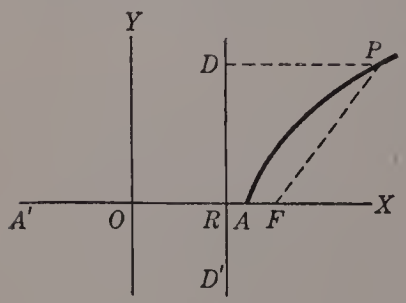
81. The hyperbola and the ellipse.—The statements and definitions just given are identical with those given on page 85 for the ellipse, except that for the ellipse e is less than unity, while for the hyperbola it is greater than unity. For this reason the hyperbola and ellipse have many properties in common as will be pointed out in detail in the following pages.



82. Value of e and the shape of the hyperbola.—We shall see that the general shape of the hyperbola is that shown in the figure and we shall see also that when e is near unity each branch is narrow and when e is very large the branches spread out so as to approach a pair of vertical lines.

83. Choice of axes.—Clearly any point may be made the focus and any line (not passing through this point) the directrix of the hyperbola, and the equation written down at once. However, as in the case of the ellipse, the equation may be simplified by a certain choice of axes.

Let $D'D$ be the directrix and F the focus of a hyperbola. Through F pass a line perpendicular to $D'D$ meeting it in R. Then on FR there is a point A between F and R such that $FA : RA = e$, and there is also a point A' on FR extended such that $FA' : RA' = e$. Hence the two points A' and A are on the hyperbola. These points are the **vertices** of the hyperbola. We take the middle point O of $A'A$, called the **center** of the hyperbola, as the origin, and the line $A'A$ as the x-axis of the coördinate system.



84. Equation of the hyperbola.—Denote the distance OA by a.

Then $$e = \frac{FA}{RA} = \frac{OF - a}{a - OR}, \quad \text{or} \quad e \cdot OR - ea = a - OF \qquad (1)$$

and $$e = \frac{FA'}{RA'} = \frac{a + OF}{OR + a}, \quad \text{or} \quad e \cdot OR + ea = a + OF. \qquad (2)$$

Adding (1) and (2), $2e \cdot OR = 2a$ or $OR = a/e$.

Subtracting (1) from (2), $2OF = 2ea$ or $OF = ea$.

Hence the coördinates of F are $(ae, 0)$ and the equation of the directrix is $x = a/e$.

Let $P(x, y)$ be any point on the hyperbola. Its distance from $F(ae, 0)$ is $\sqrt{(x - ae)^2 + y^2}$ and its distance from the directrix $x = a/e$ is $x - a/e$.

Hence by the definition of the hyperbola

$$\frac{\sqrt{(x - ae)^2 + y^2}}{x - a/e} = e.$$

Then, $$x^2 - 2aex + a^2e^2 + y^2 = e^2\left(x^2 - 2\frac{ax}{e} + \frac{a^2}{e^2}\right),$$

or $$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

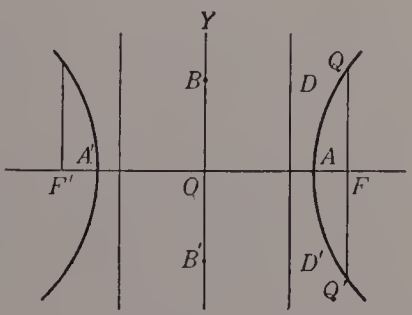
Dividing by $a^2(1 - e^2)$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1 - e^2)} = 1.$

Since $e > 1$ it follows that $1 - e^2$ is negative and hence $a^2(1 - e^2)$ is negative. The quantity $a^2(1 - e^2)$ is denoted by $-b^2$.

Then $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad [33]$$

This is the standard form of the equation of the hyperbola.

85. Directrices, foci, latus rectum.—It is an immediate consequence of the form of the equation of the hyperbola that it is symmetrical with respect to both axes. Hence the line $x = -\dfrac{a}{e}$ and the point $(-ae, 0)$ may be used as the directrix and focus respectively, and the hyperbola has the two foci $(\pm ae, 0)$ and the two directrices $x = \pm \dfrac{a}{e}$.



The line $A'A$ is the **transverse axis** of the hyperbola, and a line $B'B$ on the y-axis such that $B'O = OB = b$ is the **conjugate axis.** The double ordinate $Q'Q$ through a focus is the **latus rectum.** Its length is found by putting $x = ae$ in the equation of the hyperbola.

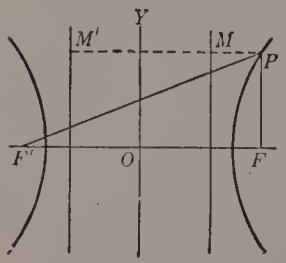
Then, $y^2 = b^2(e^2 - 1) = \dfrac{-b^2a^2(1 - e^2)}{a^2} = \dfrac{b^4}{a^2}$, and $y = \pm b^2/a$.

Hence $$Q'Q = 2b^2/a.$$

If the equation of the hyperbola is in the form $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, then the transverse axis is on the y-axis and b is the semi transverse axis while a is the semi conjugate axis.

Pages 104 and 105 should be compared carefully with §§68, 70 on pages 86 and 88 of chapter VI. The similarity of the development of the ellipse and the hyperbola will be clearly apparent.

86. A fundamental property of the hyperbola.—Let $P(x, y)$ be any point on the hyperbola.

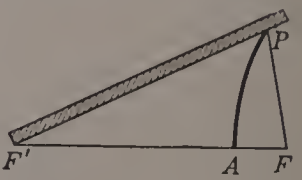


Then $$FP = e \cdot MP = e\left(x - \frac{a}{e}\right)$$
$$= ex - a,$$
and $$F'P = e \cdot M'P = e\left(x + \frac{a}{e}\right)$$
$$= ex + a.$$
Hence $$F'P - FP = 2a.$$

This is an important property of the hyperbola and is often used as the basis for the following definition:

A hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is a constant.

87. Mechanical construction of the hyperbola.—Fasten one end of a ruler at one focus F' so that it may swing about that point, and to the other end fasten a string. Make the length of the string less than that of the ruler by $2a$ and fasten the free end of it to the focus F. Press the string against the ruler with a pencil at P and turn the ruler about F'.



88. Some general properties of the hyperbola.—The proofs of the following are left to the student.

1. F is the point $(\sqrt{a^2 + b^2}, 0)$, and $c = \frac{1}{a}\sqrt{a^2 + b^2}$.
2. There is no point on the curve between the lines $x = \pm a$.
3. The curve is symmetrical with respect to the origin.

In summary we have, **Eccentricity:** $e = \frac{1}{a}\sqrt{a^2 + b^2}$ [34]; **Foci:** $(\pm ae, 0)$, or $(\pm\sqrt{a^2 + b^2}, 0)$ [35]; **Directrices:** $x = \pm a/e$ [36]; **Latus rectum:** $\frac{2b^2}{a}$ [37].

EXERCISES

1. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$ passes through the point (6, 4); find the value of a.

2. The point (6, 0) is a focus of a hyperbola and the difference of the focal radii to a point on it is 8; find the equation.

3. Find the vertices, the eccentricity, the foci, and the directrices of the hyperbolas

$$\frac{x^2}{36} - \frac{y^2}{16} = 1, \quad \text{and} \quad \frac{x^2}{40} - \frac{y^2}{20} = 1.$$

4. Find the equation of the hyperbola whose foci are $(\pm 6, 0)$ and whose eccentricity is 3/2.

5. Find the equation of the hyperbola whose vertices are $(\pm 6, 0)$ and whose directrices are $x = \pm 4$.

6. Find eccentricity and directrices of the hyperbolas $6x^2 - 9y^2 = 18$, $5x^2 - 3y^2 = 15$.

7. The latus rectum of a hyperbola is 32 and the difference of the focal radii to a point on it is 8; find its equation.

8. The directrices of a hyperbola are $x = \pm 3$ and its foci are $(\pm 12, 0)$; find its equation.

9. Find the equation in standard form of a hyperbola if its latus rectum is 9 and its eccentricity is 5/4.

10. The latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{25} = 1$ is 10; find the value of a.

11. The eccentricity of a hyperbola is 5/3 and its directrices are $x = \pm 9$; find its equation.

12. The foci of the hyperbola $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$ are $(\pm 4\sqrt{5}, 0)$; find the value of b.

13. Find the equation of the hyperbola whose conjugate axis is 12 if its latus rectum is 18.

14. Find the equation in the standard form of a hyperbola whose latus rectum is 54 and whose eccentricity is 2.

15. Find the equation of the hyperbola whose foci are $(\pm 10, 0)$ and whose latus rectum is 9.

16. The lines $x = \pm 4$ are the directrices of a hyperbola which passes through the point $(8, 2\sqrt{5})$; find its equation.

89. Tangents with given slope.—It is easy to show that for any one line tangent to a hyperbola there is another tangent parallel to it. That is, if the equation of one tangent is $y = mx + c$, then there is a number c' such that $y = mx + c'$ is also a tangent.

Eliminating y in the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad y = mx + c,$$

we have $x^2(a^2m^2 - b^2) + 2a^2cmx + a^2(c^2 + b^2) = 0.$

To make the line tangent to the hyperbola we have

$$(2a^2cm)^2 - 4(a^2m^2 - b^2)a^2(c^2 + b^2) = 0,$$

or $$4a^4c^2m^2 - 4a^4c^2m^2 - 4a^4b^2m^2 + 4a^2b^2c^2 + 4a^2b^4.$$

Hence, $$c = \pm \sqrt{a^2m^2 - b^2}$$

and the equations of the tangents with slope m are

$$y = mx + \sqrt{a^2m^2 - b^2}, \quad y = mx - \sqrt{a^2m^2 - b^2} \qquad [38]$$

Notice that $a^2m^2 - b^2$ cannot be negative, for in that case the tangent would be imaginary. Hence

$$a^2m^2 - b^2 \geqq 0 \quad \text{and} \quad a^2m^2 \geqq b^2$$

and therefore m cannot lie between $-b/a$ and b/a.

In case $m = \pm b/a$ the expression under the radical is zero and the two tangents coincide. For a study of this case see §93.

SIGHT WORK

1. Find the equations of the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, whose slopes are unity.

2. Attempt to find equations of tangents to the hyperbola $\frac{x^2}{36} - \frac{y^2}{25} = 1$, which are perpendicular to the line $3x - y + 8 = 0$; interpret result.

3. Find the equations of the tangents to the hyperbola in Ex. 2 which are parallel to the line $3x - y + 8 = 0$.

90. Tangent at a given point.—The equation of the tangent to a hyperbola at a given point on it is found as in the case of the ellipse.

By §89, $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent to the hyperbola.

If (x_1, y_1) is the point of tangency, then

$$y_1 = mx_1 + \sqrt{a^2m^2 - b^2} \tag{1}$$

and it becomes necessary to find the value of m.

From (1) $\quad y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2m^2 - b^2,$

or $\quad m^2(a^2 - x_1^2) + 2mx_1y_1 - b^2 - y_1^2 = 0,$

and $\quad m = \dfrac{-2x_1y_1 \pm \sqrt{4x_1^2y_1^2 + 4(a^2 - x_1^2)(b^2 + y_1^2)}}{2(a^2 - x_1^2)}.$

If (x_1, y_1) is on the hyperbola, then $b^2x_1^2 - a^2y_1^2 - a^2b^2 = 0$. This makes the expression under the radical equal to zero and hence,

$$m = \frac{x_1y_1}{x_1^2 - a^2}.$$

Using $x_1^2 - a^2 = \dfrac{a^2y_1^2}{b^2}$, this value of m reduces to $\dfrac{b^2x_1}{a^2y_1}$.

Hence the tangent is

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1),$$

which reduces to

$$b^2x_1x - a^2y_1y = a^2b^2 \quad \text{or} \quad \frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1 \tag{39}$$

Compare this equation with the equation of the tangents to the circle and ellipse obtained in §§ 51 and 76.

EXERCISES

1. Find the equations of the tangents to the hyperbola $9x^2 - 36y^2 = 324$ at the points for which $x = 8$.

2. Find the equations of the tangents to the hyperbola in Example 1 at the points for which $y = -3$.

91. Angle formed by tangent and normal with focal radii.— The slope of the normal to the hyperbola at the point (x_1, y_1) is $-\dfrac{a^2y_1}{b^2x_1}$ (see §90 and also §38).

The slopes of the focal radii are, $\dfrac{y_1}{x_1 + ae}$ and $\dfrac{y_1}{x_1 - ae}$.

Hence by § 39 the tangents of the angles formed by the normal and the focal radii are

$$\frac{\dfrac{y_1}{x_1 + ae} + \dfrac{a^2y_1}{b^2x_1}}{1 - \dfrac{a^2y_1^2}{b^2x_1(x_1 + ae)}} \text{ and } \frac{\dfrac{y_1}{x_1 - ae} + \dfrac{a^2y_1}{b^2x_1}}{1 - \dfrac{a^2y_1^2}{b^2x_1(x_1 + ae)}}$$

Using $e^2a^2 = a^2b^2$ and $b^2x_1^2 - a^2y_1^2 = a^2b^2$, these reduce to $\dfrac{eay_1}{b^2}$ and $-\dfrac{eay_1}{b^2}$.

Hence the normal makes equal angles with the focal radii $\tan \alpha = -\tan(-\alpha)$.

Since the tangent makes right angles with the normal, it also makes equal angles with the focal radii, namely angles whose tangents are $\dfrac{b^2}{eay_1}$ and $-\dfrac{b^2}{eay_1}$. (see (6) § 251).

These properties of the tangent and the normal to the hyperbola may also be stated as follows:

The tangent and the normal to a hyperbola at a point are the bisectors of the angles formed by the focal radii drawn to this point (compare the ellipse, § 77).

SIGHT WORK

1. Use §91 to construct a tangent and a normal to a hyperbola at a point on it when its foci are given.

2. In the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$ focal radii are drawn to the point $(8, 2\sqrt{3})$; find the angles formed by these radii and the tangent at this point.

3. Find the angles formed by the normal and the focal radii in Ex. 2.

EXERCISES

All hyperbolas in these exercises leave their axes on the coördinate axes.

1. Find the equations of the tangents to the hyperbola $5x^2 - 9y^2 = 90$ which are parallel to the line $2y - 3x + 7 = 0$.

2. Find the equations of the tangents to the hyperbola $8x^2 - 3y^2 = 72$ which are perpendicular to the line $5x - 12y + 18 = 0$.

3. Find the equations of tangents to the hyperbola $6x^2 - 18y^2 = 144$ if the slope of the tangents is 2. Explain the meaning of the result obtained by attempting to find a tangent to this hyperbola with slope $\frac{1}{3}$.

4. Find the equation of the tangent to the hyperbola $3x^2 - 8y^2 = 144$ at the point $(6\sqrt{2}, 3)$.

5. If $a = 10$, find the equation of the hyperbola whose latus rectum is equal to the semi-transverse axis.

6. Find the equation of the hyperbola which passes through the point (6, 3) if its eccentricity is 3/2.

7. Find the equation of the hyperbola whose vertices are midway between the center and the foci if the latus rectum is 12.

8. Find the equation of the hyperbola of which $x = 8$ is a directrix and (12, 0) the corresponding focus.

9. Find the equation of the hyperbola whose foci are $(\pm 10, 0)$ and whose latus rectum is $10\sqrt{2}$.

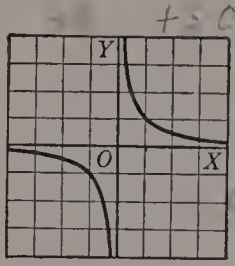
10. Find the tangents of the angles formed by the normal and the focal radii to the point $(7\sqrt{2}, 2)$ of the hyperbola, $2x^2 - 7y^2 = 168$. Also find the tangents of the angles formed by these radii and the tangent to the hyperbola at this point.

11. In the hyperbola $16x^2 - 9y^2 = 144$, find the vertices, the ends of one latus rectum, and the points whose abscissas are $\pm\sqrt{18}$ and ± 6. Sketch the hyperbola.

12. What is the smallest absolute value of y for which the values of x are not imaginary in the equation $16y^2 - 9x^2 = 144$? Find the values of x when y has values $\pm\sqrt{18}$, ± 5, and sketch the hyperbola.

13. By how much must the length of the ruler exceed that of the string in order to construct by the method of §87 the hyperbola whose equation is $25x^2 - 36y^2 = 900$? Construct this hyperbola by that method.

14. The hyperbola $b^2x^2 - 16y^2 = 16b^2$, and the ellipse $25x^2 + a^2y^2 = 25a^2$ have the common foci $(\pm 6, 0)$; find the values of a and b. Then find their points of intersection. Prove that the tangents at each of these points are perpendicular to each other.



92. Asymptotes.—Consider the curve $y = \frac{1}{x}$. As a point on the curve recedes indefinitely toward the right it approaches more and more closely to the x-axis. That is, its ordinate approaches zero. The same is true of the left branch of the curve.

As x approaches zero, y grows large indefinitely in either the positive or negative direction, and the curve approaches the y-axis.

In this case the x-axis is said to be an asymptote of the curve as is also the y-axis.

In general, if an infinite branch of a curve approaches a fixed straight line as a limit, then this line is said to be an **asymptote** of the curve.

93. Asymptotes of the hyperbola.—We will now prove that the lines

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x \tag{1}$$

are asymptotes of the hyperbola

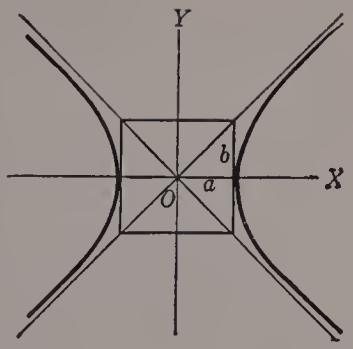
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{2}$$

From (2) $$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}.$$

It is easily seen that the line $y = \frac{b}{a}x$ is above the curve $y = \frac{b}{a}\sqrt{x^2 - a^2}$ for all positive values of x, since for any value of x the difference of the ordinates is $\frac{b}{a}(x - \sqrt{x^2 - a^2})$. Moreover, as x becomes indefinitely large, $x - \sqrt{x^2 - a^2}$ approaches zero as a limit, and hence the curve approaches the line.

The completion of the argument for x negative, and for the line $y = -\frac{b}{a}x$, is left to the student.

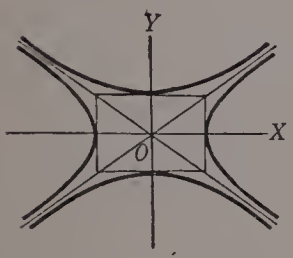
94. Construction of asymptotes.—The asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are constructed by constructing, as shown in the figure, a rectangle with sides $2a$ and $2b$ and drawing the diagonals.



95. Conjugate hyperbolas.—The two hyperbolas whose equations are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1), \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \quad (2)$$

are called **conjugate hyperbolas,** each hyperbola being the conjugate of the other. They are related as shown in the figure. The transverse axis of each of these hyperbolas is the conjugate axis of the other. It is evident from the construction of the asymptotes that the two conjugate hyperbolas have the same asymptotes.



Obviously, in the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ the vertices and foci are on the y-axis and the directrices are parallel to the x-axis. Proceeding as in § 85 we obtain: Eccentricity: $e = \frac{1}{b}\sqrt{a^2+b^2}$; Foci: $(0, \pm be)$, or $(0, \pm\sqrt{a^2 + b^2})$; Directrices: $y = \pm b/e$; Latus rectum: $2a^2/b$.

EXERCISES

1. Write the equations of the asymptotes of the hyperbola $\frac{x^2}{49} - \frac{y^2}{36} = 1$.

2. Find the ratio of the eccentricities of the two conjugate hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

3. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point (20, 12) and its conjugate hyperbola passes through $(5\sqrt{2}, 6)$; find the values of a and b.

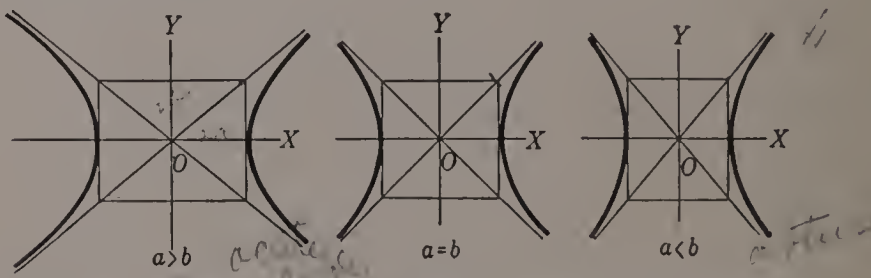
96. The rectangular hyperbola. — If in the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, a and b are equal, the hyperbola is said to be **rectangular.** In this case the slopes of the asymptotes are 1 and -1 and hence the asymptotes are at right angles to each other.

97. Acute-angled and obtuse-angled hyperbolas.—If $b < a$, then the slopes of the asymptotes lie between -1 and 1 and the hyperbola lies within the acute angle formed by the asymptotes. In this case the hyperbola is said to be **acute-angled.**

If $b > a$, the slope of one asymptote is greater than $+1$ and of the other less than -1 and the hyperbola lies within the obtuse angle formed by the asymptotes. In this case the hyperbola is said to be **obtuse-angled.**

Hence a hyperbola is acute-angled, right-angled, or obtuse-angled according as $a > b$, $a = b$, $a < b$.

From $e = \frac{1}{a}\sqrt{a^2 + b^2}$ it follows that e is large when b is great in comparison with a, while e is small (near unity) when b is small compared with a.



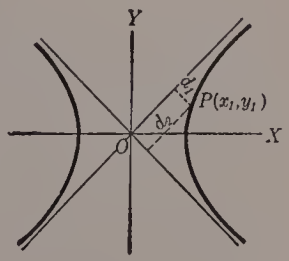
The general shape of hyperbolas of these different kinds is shown in the figure.

In case the hyperbola is rectangular, $a = b$ and $e = \frac{1}{a}\sqrt{2a^2} = \sqrt{2}$.

If the ratio of a and b remains fixed while a and b are made small indefinitely, then the hyperbola can be made to approach its asymptotes as nearly as we please.

98. Rectangular hyperbola referred to its asymptotes as axes. —If $a = b$, the equation of the hyperbola reduces to $x^2 - y^2 = a^2$, and the equations of the asymptotes are

$$x - y = 0 \quad \text{and} \quad x + y = 0.$$



Let $P(x_1, y_1)$ be any point on the hyperbola. The distances d_1 and d_2 from P to the asymptotes are, by §40,

$$d_1 = \frac{x_1 - y_1}{\sqrt{2}}, \quad d_2 = \frac{x_1 + y_1}{\sqrt{2}}.$$

Hence $d_1 d_2 = \dfrac{x_1^2 - y_1^2}{2}$.

Since (x_1, y_1) is on the hyperbola, $x_1^2 - y_1^2 = a^2$,

and $$d_1 d_2 = \frac{a^2}{2}.$$

If the asymptotes are taken as axes, d_1 and d_2 are the coördinates of the point P. Hence in this system of coördinates,

$$xy = \frac{a^2}{2} \qquad [40]$$

is the equation of the curve.

The equation $xy = k$ (k being constant) represents phenomena which occur frequently in the study of physics and chemistry. Thus Boyle's law is that under certain conditions the product of the pressure and the volume of a certain quantity of gas is a constant. This is written $pv = k$ and is represented graphically by the rectangular hyperbola referred to its asymptotes as axes.

SIGHT WORK

1. Determine which of the following hyperbolas are acute-angled, which are rectangular, and which are obtuse-angled: $8x^2 - 9y^2 = 48$, $3x^2 - 7y^2 = 42$, $9x^2 - 3y^2 = 54$, $36x^2 - 16y^2 = 400$, $14x^2 - 3y^2 = 36$, $2x^2 - 9y^2 = 81$.

2. Give the equations of the asymptotes of each hyperbola in Ex. 1.

3. Give the equations of the hyperbolas which are conjugate to each of the hyperbolas of Ex. 1.

EXERCISES

Unless otherwise stated the equations of the hyperbolas in these exercises are in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Group A

1. Find the equation of a hyperbola if the distance between its foci is 12 and the distance between its vertices is 8.

2. Find the equation of a hyperbola if the distance between its vertices is 10 and the distance between its directrices is 6.

3. In the hyperbola $9x^2 - 4y^2 = 36$ find the equation of the tangent and the normal at that extremity of a latus rectum which lies in the first quadrant.

4. Find the equations of the tangents to the hyperbola $16x^2 - 9y^2 = 144$ which pass through the point (2, 0).

5. Find the equation of the hyperbola whose asymptotes are $2x \pm y = 0$ and which passes through the point (3, 5).

6. Find the equation of the hyperbola which passes through the points (8, 6) and (10, 12).

7. Find the equation of the hyperbola whose foci are $(\pm 4, 0)$ and which passes through the point $(6, 2\sqrt{2})$.

8. Find the equation of the hyperbola whose directrices are $x = \pm 5$ and whose asymptotes are $3x - y = 0$ and $3x + y = 0$.

9. Find the angles formed by the focal radii and the normal to the hyperbola $6x^2 - 4y^2 = 144$ at the point $(4\sqrt{3}, 6)$. See § 91.

10. Find the eccentricity, foci, and directrices of the hyperbola which is conjugate to $8x^2 - 4y^2 = 27$.

Group B

1. Find the equation of a hyperbola if the distance between the directrices is 8 and the distance between the vertices is 12.

2. Find the equation of a hyperbola if the distance between the vertices is 10 and the distance between the foci is 16.

3. In the hyperbola $12x^2 - 6y^2 = 72$ find the equations of the tangent and normal at that extremity of a latus rectum which lies in the second quadrant.

4. Find the equations of the tangents to the hyperbola $16x^2 - 12y^2 = 288$ which pass through the point $(-1, 0)$.

5. The line $3x - 2y = 0$ is an asymptote of a hyperbola which passes through the point (8, 6); find the equation of the hyperbola.

6. Find the equation of the hyperbola whose directrices are $x = \pm 32/5$ and whose eccentricity is 5/4.

7. Find the equation of the hyperbola whose vertices are $(\pm 4, 0)$ and whose asymptotes are $3x + 2y = 0$, $3x - 2y = 0$.

8. Find the equation of the hyperbola which passes through the points (4, 2) and (6, 8).

9. Find the equations of the tangents to the hyperbola $6x^2 - 8y^2 = 24$ which are perpendicular to the line $2x + 6y + 7 = 0$.

10. Find the equation of the hyperbola $xy = 16$ when referred to axes which coincide with the axes of the hyperbola. Also find the foci, directrices, and vertices of this hyperbola when referred to the new axes.

Group C

1. Find the equation of a hyperbola if the distance between the directrices is 6 and the distance between the foci is 12.

2. Find the equation of a hyperbola if the distance between the directrices is 12 and the distance between the vertices is 16.

3. In the hyperbola $8x^2 - 6y^2 = 96$ find the equations of the tangent and normal at that end of a latus rectum which lies in the third quadrant.

4. Find the equations of the tangents to the hyperbola $18x^2 - 12y^2 = 108$ which pass through the point (2, 0).

5. The line $5x + 4y = 0$ is an asymptote of a hyperbola whose equation is of the form $b^2x^2 - a^2y^2 = a^2b^2$; find the values of a and b if the hyperbola passes through the point $(-6, 4)$.

6. Find the equation of the hyperbola whose foci are $(\pm 5, 0)$ and whose asymptotes are $2x - y = 0$ and $2x + y = 0$.

7. Find the equation of the hyperbola whose asymptotes are $x - y = 0$ and $x + y = 0$, and whose directrices are $x = \pm 6$.

8. In the hyperbola $9x^2 - 4y^2 = 36$ find the angles between the normal and the focal radii at the point $\left(\frac{10}{3}, 4\right)$. See § 91.

9. Find the equation of the hyperbola which passes through the points $(-4, 2)$ and $(6, -8)$.

10. Find the equation of the hyperbola whose conjugate hyperbola passes through the point $(8, 3\sqrt{5})$ if the foci of the conjugate hyperbola are $(0, \pm 5)$.

99. Solution of Problems.—Further properties of the hyperbola are stated in the following examples and in the problems that follow.

Example 1. Prove that the distance from the origin (center) to a point on a rectangular hyperbola is a mean proportional between the distances from this point to the foci.

PROOF: In the rectangular hyperbola $e = \sqrt{2}$ (§97), and the foci are $F'(-a\sqrt{2}, 0)$, $F(a\sqrt{2}, 0)$. Let $P(x, y)$ be any point on the hyperbola. We need to prove that $F'P \cdot FP = \overline{OP}^2$.

The distances from $P(x, y)$ to the foci are $\sqrt{(x + a\sqrt{2})^2 + y^2}$ and $\sqrt{(x - a\sqrt{2})^2 + y^2}$ and the square of their product is

$$\begin{aligned}
&[(x + a\sqrt{2})^2 + y^2]\,[(x - a\sqrt{2})^2 + y^2] \\
&= (x^2 + y^2 + 2a^2 + 2ax\sqrt{2})\,(x^2 + y^2 + 2a^2 - 2ax\sqrt{2}) \\
&= x^4 + y^4 + 4a^4 + 2x^2y^2 + 4a^2x^2 + 4a^2y^2 - 8a^2x^2 \\
&= x^4 + y^4 + 2x^2y^2 + 4a^2(a^2 + y^2 - x^2).
\end{aligned}$$

If (x, y) is on the rectangular hyperbola, then

$$x^2 - y^2 = a^2 \quad \text{or} \quad a^2 + y^2 - x^2 = 0.$$

Hence the produet of the distances is

$$\sqrt{x^4 + y^4 + 2x^2y^2} = x^2 + y^2 = \overline{OP}^2.$$

Example 2. A line cuts a hyperbola in the points P' and P, and the asymptotes in R' and R. Show that the middle point of $P'P$ is also the middle point of $R'R$.

PROOF: Let $y = mx + c$ be the line and $b^2x^2 - a^2y^2 = a^2b^2$ the hyperbola. Then

$$b^2x^2 - a^2(mx + c)^2 = a^2b^2,$$

or

$$x^2(b^2 - a^2m^2) - 2a^2cmx - a^2c^2 - a^2b^2 = 0.$$

The roots of this equation give the abscissas of P' and P and half the sum of the roots is the abscissa of the middle point of $P'P$. But half this sum is $\dfrac{a^2cm}{b^2 - a^2m^2}$. (See § 241).

Solving $y = mx + c$ simultaneously for x with eaeh of the equations of the asymptotes, we have

$$x = \frac{ac}{b - am} \text{ and } x = -\frac{ac}{b + am} \text{ as the abscissas of } R' \text{ and } R.$$

Half the sum of these values of x is $\dfrac{a^2cm}{b^2 - a^2m^2}$.

Since only one point on the line $y = mx + c$ has this abscissa it follows that the middle points of $P'P$ and $R'R$ coincide.

Example 3. The three vertices of a triangle lie on an equilateral hyperbola; prove that the altitudes of the triangle meet on the hyperbola.

PROOF: Let the hyperbola be referred to its asymptotes as axes. Then (§98) its equation is $xy = k$. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be the vertices, and let

$$y = mx + c \tag{1}$$

be the line through P_3 which is perpendicular to P_1P_2. Substituting from $xy = k$ in (1)

$$y = \frac{mk}{y} + c, \text{ or } y^2 - cy - mk = 0. \tag{2}$$

Since (1) passes through P_3, y_3 is one root of (2) and hence $-\dfrac{mk}{y_3}$ is the other root (see § 241). That is, (1) meets the hyperbola in a point Q whose ordinate is $-mk/y_3$.

Since (1) is perpendicular to P_1P_2 and P_1 and P_2 lie on the hyperbola,

$$-\frac{mk}{y_3} = \frac{\dfrac{x_1 - x_2}{y_1 - y_2}k}{y_3} = \frac{x_1 - x_2}{y_3(y_1 - y_2)}k = \frac{\left(\dfrac{k}{y_1} - \dfrac{k}{y_2}\right)k}{y_3(y_1 - y_2)} = -\frac{k^2}{y_1y_2y_3},$$

which is the ordinate of the point Q in which (1) meets the hyperbola. It follows at once from the symmetry with which y_1, y_2, y_3 are involved in the result that the other altitudes meet the hyperbola in the same point.

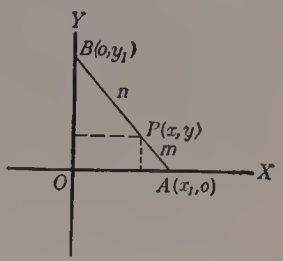
Example 4. A line segment AB moves so that its end-points are on the coördinate axes. The line segment varies in length so that the area of the triangle formed by it and the axes is constant. Find the locus of a point P on AB if $AP : PB = m : n$ is a constant ratio.

SOLUTION: Using the notation shown in the figure

$$\frac{x_1}{x} = \frac{m+n}{n}, \text{ or } x_1 = \frac{x(m+n)}{n}$$

and

$$\frac{y_1}{y} = \frac{m+n}{m}, \text{ or } y_1 = \frac{y(m+n)}{m}.$$



But $x_1y_1 = k$, a constant.

Hence $\dfrac{x(m+n)}{n} \cdot \dfrac{y(m+n)}{m} = k$, or $xy = \dfrac{mnk}{(m+n)^2}$,

whose locus is an equilateral hyperbola referred to its asymptotes as axes.

MISCELLANEOUS PROBLEMS ON CHAPTER VII

1. Find the equation of the hyperbola $8x^2 - 8y^2 = 64$ when referred to its asymptotes as axes.

2. Prove that the tangent to a hyperbola at a vertex meets the asymptotes on a circle of which the line joining the foci is the diameter.

3. If a tangent to a hyperbola is cut by the tangents at the vertices in points P and Q, show that the circle on the diameter PQ passes through the foci.

4. If an asymptote of a hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ makes an angle α with the x-axis prove that the eccentricity of the hyperbola is $\sec \alpha$.

5. Prove that the part of an asymptote which lies between the directrices is equal to the transverse axis.

6. Find the equations of the tangents to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ which are perpendicular to the asymptote $y = \frac{b}{a}x$.

7. Prove that the latus rectum of a hyperbola is a third proportional to the axes.

8. Find the equations of the tangent and normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ at a point whose ordinate and abscissa are equal. Suppose that $b > a$.

9. Find the equations of the tangents to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ from the point on the positive side of the origin where a directrix meets the x-axis.

10. Prove that the foot of the perpendicular from a focus of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ on an asymptote is at a distance a from the center and a distance b from the focus.

11. Prove that the product of the perpendiculars from any point on a hyperbola on the asymptotes is a constant.

12. Prove that if a series of hyperbolas have the same transverse axis and a line perpendicular to this axis meets these hyperbolas in the points P_1, P_2 . . . then the tangents at these points meet the transverse axis in the same point.

13. A variable circle is tangent to each of two fixed unequal circles; show that the locus of its center is an ellipse or a hyperbola.

14. Prove that a circle with one focus of $b^2x^2 - a^2y^2 = a^2b^2$ as center and b as a radius is tangent to the asymptotes at the points where they meet the corresponding directrix.

15. A line cuts a hyperbola in P' and P and cuts the asymptotes in Q' and Q; show that $Q'P' = PQ$.

CHAPTER VIII

LINEAR TRANSFORMATION OF COÖRDINATES

We have already seen that the equations

$$x^2 + y^2 = r^2 \quad \text{and} \quad (x - h)^2 + (y - k)^2 = r^2$$

both represent circles with radius r, the first having its center at the origin and the second at the point (h, k).

It appears, therefore, that by substituting $x - h$ for x and $y - k$ for y the circle is moved a distance h to the right and a distance k upward provided h and k are both positive. If h is negative, the circle is moved to the left, and if k is negative, it is moved down. Or we may say that this substitution moves the axes a distance h to the left if h is positive, and downward a distance k if k is positive. Changing the sign of h or k moves the axes in the opposite directions.

It follows that substituting $x + h$ for x moves the axes a distance h to the right or left according as h is positive or negative, and substituting $y + k$ for y causes a similar motion of the axes in the vertical direction.

This is a special case of a general proposition called the **translation theorem,** which may be stated as follows:

If in an equation in x *and* y, x *is replaced by* x + h, *and* y *is replaced by* y + k, *then the axes are moved* h *units in the horizontal direction and* k *units in the vertical direction, the motion in each case being in the positive or negative direction according as* h *and* k *are positive or negative.*

That is, if h is positive the axes are moved to the right and if h is negative they are moved to the left; similarly the axes are moved up or down according as k is positive or negative.

100. Proof of the translation theorem.—If the axes are moved a distance h to the right, the x-coördinate of each point is decreased by h; and if the axes are moved upward a distance k, the y-coördinate of each point is decreased by k. Hence if x, y are the coördinates of a point referred to the old axes and x', y' its coördinates referred to the new axes, then

$$x = x' + h \quad \text{and} \quad y = y' + k.$$

It follows that if in the equation of a figure x is replaced by $x' + h$ and y is replaced by $y' + k$, then the result is the equation of the same figure referred to a system of axes whose origin is the point (h, k) in the original system.

That is, by this substitution the axes are translated a distance (and direction) h horizontally, and a distance (and direction) k vertically, the axes remaining parallel to the original axes.

In the new equation obtained by the substitution the accents are usually suppressed. Hence we say that we substitute $x + h$ for x and $y + k$ for y.

Example 1. Find the equation of the line $2x - 3y + 7 = 0$ when referred to a new set of axes, parallel to the old, with origin at the point (2, 5).

SOLUTION: The axes are to be moved 2 units to the right and 5 units upward. We therefore substitute $x + 2$ for x and $y + 5$ for y. The result is,

$$2(x + 2) - 3(y + 5) + 7 = 0,$$

or

$$2x - 3y - 4 = 0.$$

Example 2. Find the equation of the circle $x^2 + y^2 - 4x + 5y - 10 = 0$ referred to parallel axes through the point $(5, -4)$.

SOLUTION: Substituting $x + 5$ for x and $y + (-4)$ for y, we have

$$(x + 5)^2 + (y - 4)^2 - 4(x + 5) + 5(y - 4) - 10 = 0,$$

or

$$x^2 + y^2 + 6x - 3y - 9 = 0.$$

101. Uses of translation of axes.—It is frequently possible to simplify an equation by translating the axes (or what is the same, by translating the figure). This is shown in the following examples.

Example 1. Translate the axes so that the line $2x + 7y - 10 = 0$ will pass through the new origin.

FIRST SOLUTION: Obviously this may be done by moving the origin to any point on the line. We may move the origin to the point (5, 0) by substituting $x + 5$ for x and $y + 0$ for y. The result is

$$2(x + 5) + 7y - 10 = 0, \text{ or } 2x + 7y = 0.$$

SECOND SOLUTION: Let (h, k) be the new origin. Then we substitute $x + h$ for x and $y + k$ for y, and obtain

$$2(x + h) + 7(y + k) - 10 = 0, \text{ or } 2x + 7y + 2h + 7k - 10 = 0.$$

If the line is to pass through the origin, it is necessary to select values for h and k such that $2h + 7k - 10 = 0$. This may be done in many ways. Thus if we let $k = 0$, then $h = 5$. Compare FIRST SOLUTION. Ordinarily there is an endless number of solutions.

Example 2. Translate the axes so that the new origin shall be at the center of the circle $x^2 + y^2 - 10x + 4y - 12 = 0$. That is, reduce the equation to the form $x^2 + y^2 = r^2$.

SOLUTION: Let the coördinates of the new origin be (h, k). Substituting $x + h$ for x and $y + k$ for y gives

$$(x + h)^2 + (y + k)^2 - 10(x + h) + 4(y + k) - 12 = 0,$$

or $$x^2 + y^2 + x(2h - 10) + y(2k + 4) + h^2 + k^2 - 10h + 4k - 12 = 0.$$

In the required form of the equation, the coefficients of x and y must be zero. That is, $2h - 10 = 0$ and $2k + 4 = 0$, or $h = 5$ and $k = -2$. Then the equation reduces to $x^2 + y^2 = 41$.

Since $x + 5$ is substituted for x and $y - 2$ for y, the axes are moved 5 to the right and 2 down.

EXERCISES

In each of following examples the equations represent the same curve. Compare the location of the axes.

1. $x^2 + y^2 = 40$, $(x - 4)^2 + (y - 3)^2 = 40$, and $(x + 2)^2 + (y + 1)^2 = 40$.

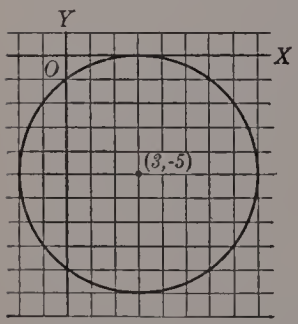
2. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, $\dfrac{(x + 2)^2}{25} + \dfrac{(y - 3)^2}{9} = 1$, and $\dfrac{(x - 6)^2}{25} + \dfrac{(y + 4)^2}{9} = 1$.

3. $y^2 = 12x$, $(y + 4)^2 = 12(x - 1)$, $(y - 3)^2 = 12(x + 5)$, and $(y + 2)^2 = 12(x + 6)$.

102. Study of general conic by completing squares.—Before studying the general method of simplifying equations by translating the axes we will learn to simplify equations of the second degree by a more special method. The examples that follow show how the properties of curves represented by the equation

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0$$

may be studied by completing squares. Essentially the method is the same as that studied in §§ 103–109 when the more general method is applied to second degree equations.



Example 1. Determine the character of the locus of $x^2 + y^2 - 6x + 10y + 6 = 0$.

SOLUTION: Writing this equation in the form

$$x^2 - 6x + y^2 + 10y = -6$$

and adding 9 and 25 to both members we have

$$x^2 - 6x + 9 + y^2 + 10y + 25 = 28,$$

or $(x - 3)^2 + (y + 5)^2 = 28$.

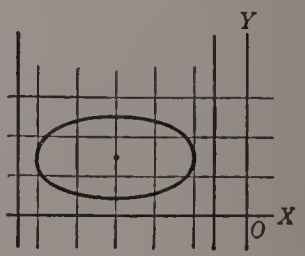
But this equation represents a circle with center $(3, -5)$ and radius $\sqrt{28}$.

Example 2. Determine the character of the locus of $x^2 + 4y^2 + 6x - 12y + 14 = 0$.

SOLUTION: Completing squares as in Example 1,

$$x^2 + 6x + 9 + 4\left(y^2 - 3y + \frac{9}{4}\right) = -14 + 9 + 9 = 4,$$

or $$(x + 3)^2 + 4\left(y - \frac{3}{2}\right)^2 = 4.$$

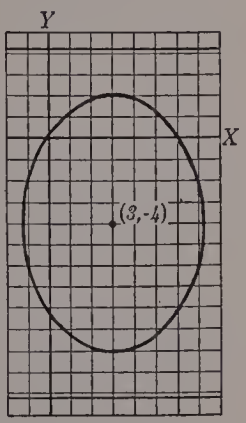


But this equation represents an ellipse with center $\left(-3, \frac{3}{2}\right)$ and semi-axes $a = 2$ and $b = 1$, and with its major axis parallel to the x-axis.

The eccentricity is $\frac{1}{2}\sqrt{2^2 - 1^2} = \frac{1}{2}\sqrt{3}$ and the distances from the center to the foci and the directrices are $ae = \sqrt{3}$ and $\frac{a}{e} = 2/\frac{1}{2}\sqrt{3} = \frac{4}{3}\sqrt{3}$.

Hence the foci are $(-3 + \sqrt{3}, 3/2)$, $(-3 - \sqrt{3}, 3/2)$ and the directrices are $x = -3 - \frac{4}{3}\sqrt{3}$, $x = -3 + \frac{4}{3}\sqrt{3}$.

Example 3. Determine the character of the locus $2x^2 + y^2 - 12x + 8y - 2 = 0$.



SOLUTION: Completing squares as before

$$2(x^2 - 6x + 9) + (y^2 + 8y + 16) = 36,$$

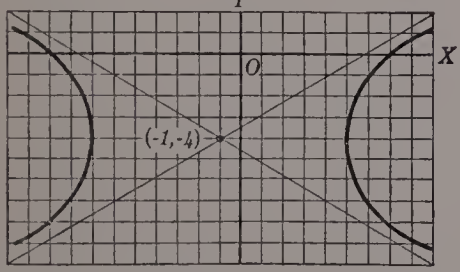
$$\text{or } 2(x - 3)^2 + (y + 4)^2 = 36.$$

But this equation represents an ellipse with center $(3, -4)$ and semiaxes $a = 3\sqrt{2}$, $b = 6$. Since $a < b$, the major axis is parallel to the y-axis.

The eccentricity is $\frac{1}{6}\sqrt{36 - 18} = \frac{1}{2}\sqrt{2}$, and the distances from the center to the foci and the directrices are $3\sqrt{2}$ and $6\sqrt{2}$ respectively.

Hence the foci are $(3, -4 + 3\sqrt{2})$, $(3, -4 - 3\sqrt{2})$ and the directrices are $y = -4 + 6\sqrt{2}$ and $y = -4 - 6\sqrt{2}$.

Example 4. Determine the character of the locus of $4x^2 - 12y^2 + 8x - 48y - 188 = 0$.



SOLUTION: Completing squares,

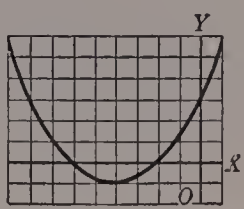
$$4(x^2 + 2x + 1) - 12(y^2 + 4y + 4) = 188 + 4 - 48 = 144,$$

$$\text{or } \frac{(x + 1)^2}{36} - \frac{(y + 2)^2}{12} = 1.$$

But this equation represents a hyperbola with center $(-1, -2)$ and semiaxes $a = 6$, $b = 2\sqrt{3}$.

The eccentricity is $\frac{1}{a}\sqrt{a^2 + b^2} = \frac{2}{3}\sqrt{3}$, and the distances from the center to the foci and the directrices are $ae = 4\sqrt{3}$ and $\frac{a}{e} = 3\sqrt{3}$.

Hence the foci are $(-1 + 4\sqrt{3}, -2)$, $(-1 - 4\sqrt{3}, -2)$ and the directrices are $x = -1 + 3\sqrt{3}$ and $x = -1 - 3\sqrt{3}$.



Example 5. Determine the character of the locus of $x^2 - 4y + 8x + 12 = 0$.

SOLUTION: Completing the square in $x^2 + 8x$,

$$x^2 + 8x + 16 - 4y = 4, \quad \text{or} \quad (x + 4)^2 - 4y = 4.$$

But this equation represents a parabola whose axis is $x = -4$ and whose lowest point, the vertex, is $(-4, -1)$.

103. Eliminating first-degree terms.—We will now study a more general method for simplifying equations by translating the axes.

Consider $$Ax^2 + By^2 + 2Dx + 2Ey + C = 0. \tag{1}$$

Substituting $x + h$ for x and $y + k$ for y, we obtain

$$A(x + h)^2 + B(y + k)^2 + 2D(x + h) + 2E(y + k) + C = 0,$$

or $$Ax^2 + By^2 + 2x(Ah + D) + 2y(Bk + E) + Ah^2 + Bk^2 + 2Dh + 2Ek + C = 0. \tag{2}$$

To eliminate the first-degree terms it is necessary that $Ah + D = 0$ and $Bk + E = 0$, or $h = -D/A$ and $k = -E/B$.

Then (2) reduces to

$$Ax^2 + By^2 = \frac{BD^2 + AE^2 - ABC}{AB}. \tag{3}$$

Note that in this transformation the coefficients of x^2 and y^2 are not changed.

Equation (3) represents the same figure as does equation (1), and hence the character of the locus of (1) may be determined by considering equation (3).

104. The imaginary locus. — For simplicity, represent $\dfrac{BD^2 + AE^2 - ABC}{AB}$ in equation (3) of the preceding by K. If A and B are both positive, and if $BD^2 + AE^2 - ABC$ is negative, then equation (3) reduces to $Ax^2 + By^2 = K$ where K is a negative number. This equation is satisfied by no real values of x and y, and we say that the locus is **imaginary.**

105. The circle.—If $A = B$, both being positive, then the general equation reduces to

$$x^2 + y^2 = \frac{1}{A} \cdot \frac{BD^2 + AE^2 - ABC}{AB} = \frac{K}{A}.$$

If K is positive the locus is a circle with radius $\sqrt{\dfrac{K}{A}}$.

106. The ellipse and the hyperbola; straight lines.—If A, B, and K are all positive and A is different from B, then $Ax^2 + By^2 = K$ reduces to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is an ellipse whose semiaxes are $a = \sqrt{\frac{K}{A}}$ and $b = \sqrt{\frac{K}{B}}$.

If one of the coefficients A and B is positive and the other negative, the equation $Ax^2 + By^2 = K$ represents a hyperbola whether K is positive or negative, since the equation reduces to one of the forms

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

If eliminating the first degree terms as in §103 reduces the equation to $Ax^2 + By^2 = 0$; that is, if $BD^2 + AE^2 - ABC = 0$, and if A and B have different signs, then by factoring we obtain an equation representing two straight lines (see §45). If in $Ax^2 + By^2 = 0$, A and B have the same sign and $K = 0$, the locus is a single point, namely the origin.

107. The center and axes of central conics.—The circle, the ellipse, and the hyperbola are the central conics. It follows from the preceding paragraphs that if in the equation

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0$$

A and B are different from zero, and if the equation has a real locus, then this locus is a central conic, two straight lines, or a point.

The axes of the conic represented by this equation are parallel to the coördinate axes. From the substitution in §103 it follows that the center of the conic is $-D/A, -E/B$ and its axes are on the lines $x = -D/A$, $y = -E/B$.

If it is given that the axes of the conic are on the lines $x = h$, $y = k$, and if a and b are its semiaxes, then its equation is

$$\frac{(x-h)^2}{a^2} \pm \frac{(y-k)^2}{b^2} = 1.$$

108. The parabola.—If A or B is equal to zero (but not both), then only one of the first-degree terms in the equation

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0 \qquad (1)$$

can be removed by the substitution of §103.

If $A = 0$ and $B \neq 0$, the substitution $x = x + h$, $y = y + k$ reduces this equation to

$$By^2 + 2Dx + (2Bk + 2E)y + Bk^2 + 2Dh + 2Ek + C = 0.$$

We may then find values of h and k so that the coefficient of y and the constant part of the equation shall both be zero. That is, we put

$$2Bk + 2E = 0 \quad \text{and} \quad Bk^2 + 2Dh + 2Ek + C = 0,$$

which gives, $\qquad k = -\frac{E}{B}, \; h = \frac{E^2 - BC}{2BD}.$

Then we have $By^2 + 2Dx = 0$, or $y^2 = -2Dx/B$ which represents a parabola for all real values of B and D which are not zero.

If $B = 0$ and $A \neq 0$, equation (1) reduces to

$$x^2 = -2Ey/A$$

which is likewise the equation of a parabola.

In case $A = 0$ and $B \neq 0$, $\left(\frac{E^2 - BC}{2BD}, -\frac{E}{B}\right)$ is the vertex and $y = -\frac{E}{B}$ is the axis of the parabola represented by (1).

In case $A \neq 0$ and $B = 0$, $\left(-\frac{D}{A}, \frac{D^2 - AC}{2AE}\right)$ is the vertex and $x = -D/A$ is the axis.

If it is given that the point (h, k) is the vertex of a parabola and that its axis is on the line $y = k$, then its equation is

$$(y - k)^2 = 4p(x - h).$$

109. Summary.—As a summary of §§ 103–108 we now have:

(a) Translating the axes does not change the coefficients of x^2 and y^2 in the equation

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0 \tag{1}$$

If $A \neq 0$, $B \neq 0$, (1) may be reduced to

$$Ax^2 + By^2 = \frac{BD^2 + AE^2 - ABC}{AB} \tag{2}$$

by translating axes.

(b) If $A = B$ and if the locus is real, then it is a circle.

(c) If $A \neq B$, both being positive, and if the locus is real, then it is an ellipse.

(d) If A and B are both different from zero and different in sign, then the locus is always a real hyperbola.

(e) If (1) represents a central conic, then the center of the conic is $\left(-\frac{D}{A}, -\frac{E}{B}\right)$.

EXERCISES

In the following eliminate all first-degree terms which can be removed by translating the axes, and state what kind of locus is represented by each equation. Plot the curves showing both sets of axes.

Group A

1. $x^2 + y^2 - 2x + 4y - 10 = 0$
2. $y^2 + 4y - 6x - 8 = 0$
3. $x^2 + 4y^2 + 4x - 8y - 8 = 0$
4. $2x^2 + 5y^2 + 4x + 20y + 22 = 0$
5. $x^2 + y^2 + 4x - 6y + 15 = 0$
6. $x^2 - 9y^2 - 2x + 36y + 1 = 0$
7. $y^2 + 8x + 6y + 1 = 0$
8. $x^2 - y^2 + 8x + 6y + 7 = 0$
9. $4x^2 + 9y^2 + 16x - 54y + 169 = 0$
10. $x^2 + y^2 - 5x + 7y + 4 = 0$
11. $3x^2 + 5y^2 - 12x + 10y - 43 = 0$
12. $8x^2 - 3y^2 + 6x - 12y + 6 = 0$

Group B

1. $3x^2 + y^2 + 12x - 2y + 13 = 0$
2. $x^2 + 8x + y^2 - 6y + 24 = 0$
3. $y^2 + 8y - 8x + 32 = 0$
4. $25x^2 + 9y^2 - 150x + 18y + 9 = 0$
5. $4x^2 - 9y^2 + 8x + 18y - 5 = 0$
6. $x^2 + y^2 - 6x - 8y + 19 = 0$
7. $y^2 + 6x - 6y - 9 = 0$
8. $9x^2 + 4y^2 - 72x + 8y + 292 = 0$
9. $25x^2 - 16y^2 + 100x + 96y - 444 = 0$
10. $5x^2 + 5y^2 + 12x - 30 = 0$
11. $5x^2 + 2y^2 - 8x + 10y - 60 = 0$
12. $5x^2 - 12y^2 + 8x - 4y + 25 = 0$

Group C

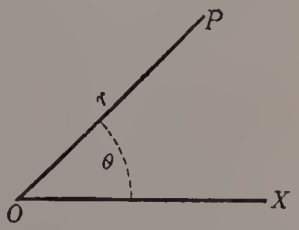
1. $y^2 - 12x - 6y - 15 = 0$
2. $x^2 + y^2 - 10y - 4x - 35 = 0$
3. $x^2 + 4y^2 - 2x - 24y + 37 = 0$
4. $x^2 + y^2 - 6x - 8y + 26 = 0$
5. $y^2 + 4y + 10x - 26 = 0$
6. $16x^2 + 25y^2 - 100y - 128x - 44 = 0$
7. $25x^2 - y^2 - 150x + 2y + 224 = 0$
8. $16x^2 + 25y^2 - 32x - 50y + 441 = 0$
9. $4x^2 + 4y^2 - 9x - 12y - 6 = 0$
10. $49x^2 - 4y^2 + 98x + 16y - 163 = 0$
11. $7x^2 - 2y^2 - 14x + 8y - 8 = 0$
12. $16x^2 + 36y^2 + 8x - 6y - 125 = 0$

CHAPTER IX

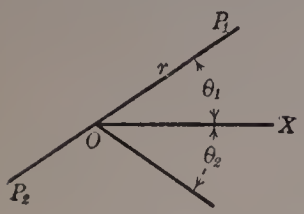
POLAR COÖRDINATES

The rectangular coördinates of a point specify its location by giving its directions and distances from two intersecting straight lines called the axes.

What are called **polar coördinates** are used to specify the location of a point by giving its distance and direction from a fixed point. This fixed point, called the **origin** or **pole**, is denoted by O in the figure. A fixed line OX, called the **initial line** or **axis**, is used for the purpose of fixing a direction. The location of the point P is specified by giving the distance $OP = r$ and the angle $XOP = \theta$. The quantities r, θ are called **polar coördinates** of the point P; r is called the **radius vector** of the point and θ is called its **amplitude**.



110. Signs of polar coördinates.—The angle θ has its initial line on the fixed line OX and is regarded as positive when taken in the counter-clockwise direction.



Thus in the figure, θ_1 is positive and θ_2 is negative.

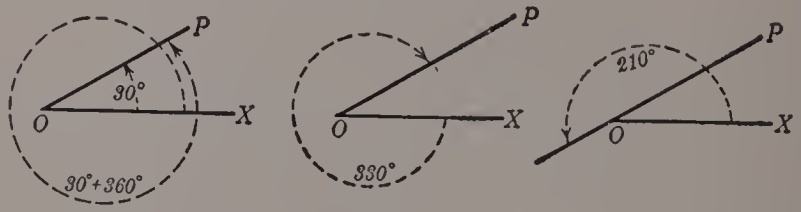
When the radius vector is measured from O along the terminal line of θ it is regarded as positive, and when measured in the opposite direction it is negative.

Thus in the figure the coördinates of P_1 are r, θ, and the coördinates of P_2 may be taken as $-r$, θ.

111. Polar coördinates of a point not unique.—In rectangular coördinates only one set of coördinates can represent a point. In polar coördinates, however, one point may be represented by many sets of coördinates.

Suppose, for example, that the coördinates are 2, 30°.

This point may also be represented by (2, 360° + 30°), or (2, 390°); by (2, 720° + 30°), or (2, 750°); etc.

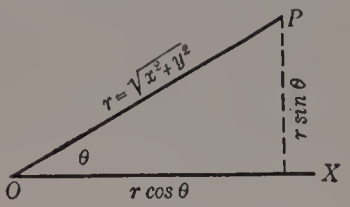


The same point may further be represented by (2, − 330°); by (2, − 360° − 330°), or (2, − 690°); etc.

Again (− 2,210°); (− 2, 360° + 210°), or (− 2, 570°) represent the same point as do also (− 2, − 150°); (−2, − 150° − 360°), or (− 2, − 510°); etc.

The student should make clear to himself the various ways of representing the same point as shown in the cases just described.

112. Relation between polar and rectangular coördinates.—If in a rectangular and a polar system of coördinates the origins of the two systems coincide and the polar axis of the polar system lies along the positive part of the x-axis, and if x, y and r, θ are the coördinates of the same point, then



$$x = r\cos\theta, \quad y = r\sin\theta \qquad [41]$$

and also

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x} \qquad [42]$$

From these relations we find,

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad [43]$$

By means of these relations an equation may be changed from either system of coördinates to the other.

Example 1. Transform $2x - 3y + 6 = 0$ into polar coördinates.

SOLUTION: Substituting from [41], we have

$$2r\cos\theta - 3r\sin\theta + 6 = 0,$$
$$r(2\cos\theta - 3\sin\theta) + 6 = 0.$$

Example 2. Transform $x^2 + y^2 = a^2$ into polar coördinates.

SOLUTION: Substituting $x^2 = r^2\cos^2\theta$, $y^2 = r^2\sin^2\theta$ we have

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2.$$

Hence
$$r^2(\cos^2\theta + \sin^2\theta) = a^2,$$

which reduces to $r^2 = a^2$, or $r = a$.

Example 3. Transform $x^2 + y^2 - 4x + 6y - 10 = 0$ into polar coördinates.

SOLUTION: Substituting as above, we have

$$r^2\cos^2\theta + r^2\sin^2\theta - 4r\cos\theta + 6r\sin\theta - 10 = 0,$$
$$r^2 - r(4\cos\theta - 6\sin\theta) - 10 = 0.$$

EXERCISES

1. Give six different sets of coördinates of the point (5, 60°), three in which the value of r is positive and three in which it is negative.

2. Show that $(r, \theta + 2n\pi)$ represents the same point for all integral values of n, including $n = 0$.

3. Give six different sets of coördinates of the point $(-3, -120°)$, three in which the value of θ is positive and three in which it is negative.

4. Show that $[r, \theta + (2n+1)\pi]$ and $(-r, \theta + 2n\pi)$ represent the same point for all integral values of n, including $n = 0$.

5. Find polar coördinates of each of the points (3, 7), (5, − 6), (− 2, 7), (− 3, − 1). Make r and θ both positive in each case.

6. Find the rectangular coördinates of the points (2, 30°), (3, 60°), (5, 120°), (4, 240°).

7. Transform each of the following equations into polar coördinates: $x - 7y - 3 = 0$, $x^2 + y^2 - 9x + 7y - 16 = 0$, $x^2 + y^2 + 7x - 18 = 0$.

8. Transform the following equations into polar coördinates: $y^2 = 4px$, $x^2 = 4py$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

113. Symmetry.—The proofs of the following propositions are left to the student.

(a) *If a polar equation remains unchanged by substituting* $-\theta$ *for* θ *the locus is symmetrical with respect to the polar axis.*

(b) *If a polar equation remains unchanged by substituting* $\pi - \theta$ *for* θ *the locus is symmetrical with respect to the perpendicular to the polar axis at the origin.*

(c) *If a polar equation remains unchanged by substituting* $-r$ *for* r *the locus is symmetrical with respect to the pole.*

Thus, the locus of a polar equation in which θ occurs only in $\cos\theta$ or $\sec\theta$, or in both, is symmetrical with respect to the polar axis, since $\cos\theta = \cos(-\theta)$ and $\sec\theta = \sec(-\theta)$.

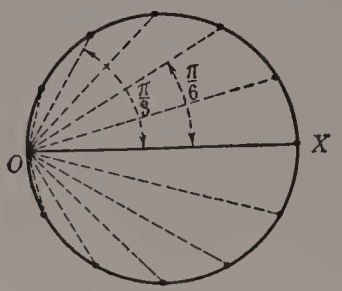
Again, the locus of a polar equation in which θ occurs only in $\sin\theta$ or in $\csc\theta$, or in both, is symmetrical with respect to the normal to the polar axis at the origin, since $\sin\theta = \sin(\pi - \theta)$ and $\csc\theta = \csc(\pi - \theta)$.

Example. Trace the curve $r = 4\cos\theta$, and discuss as to symmetry.

SOLUTION: This equation is satisfied by the following pairs of values.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
r	4	$2\sqrt{3}$	2	0	-2	$-2\sqrt{3}$	-4	$-2\sqrt{3}$	-2	0	2	$2\sqrt{3}$	4

Explain in detail how each point given in the table is plotted,



(a) for values of θ from 0 to $\frac{\pi}{2}$,

(b) for values of θ from $\frac{\pi}{2}$ to $\frac{3}{2}\pi$,

(c) for values of θ from $\frac{3}{2}\pi$ to 2π.

Discuss this locus as to symmetry.

Also consider the locus of this equation using the theorem that a right angle may be inscribed in a semicircle.

EXERCISES

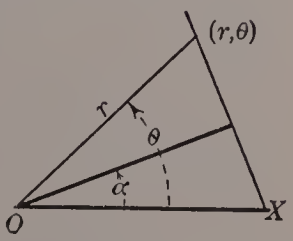
Discuss as to symmetry the locus of each of the following equations.

1. $r = a \cos\theta$
2. $r = a \sin\theta$
3. $r = a \sec\theta$
4. $r = a \operatorname{esc}\theta$
5. $r^2 = a \cos\theta$
6. $r^2 = a \cos^2\theta$
7. $r = \dfrac{2p}{1 - e\cos\theta}$
8. $r^2 = 2a \tan^2\theta$
9. $r^2 = a \sin\theta$
10. $r^2 = a \sin^2\theta$
11. $r = 4a \cos^3\theta$
12. $r = 4a \sin^3\theta$

114. Polar equation of the straight line.—Let p be the perpendicular distance from the origin to a straight line and α the angle which this perpendicular makes with the initial line. Then for any point (r, θ) on the line,

$$r \cos(\theta - \alpha) = p,$$

which is the required equation.



Example 1. Find the equation of the line through the pole and the point $\left(2, \frac{\pi}{3}\right)$.

SOLUTION: Since θ is a constant (the same for all points of the line), the equation of the line is $\theta = \dfrac{\pi}{3}$.

Example 2 Find the equation of the line through $(a, 0)$ which is perpendicular to the polar axis.

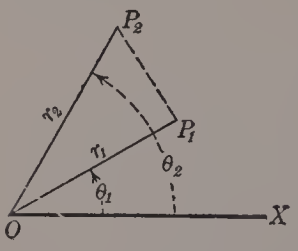
SOLUTION: Show that the equation is $r = a \sec\theta$.

EXERCISES

1. Find the equation of the line through $\left(a, \frac{\pi}{2}\right)$ which is parallel to the polar axis.

2. The normal from the pole to a line is 8 in length and makes an angle of $\frac{\pi}{4}$ with the polar axis; write the equation of the line.

115. Distance between two points in terms of polar coordinates.



Let $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ be any two points in the plane.

Then $OP_1 = r_1$ and $OP_2 = r_2$.

Also $\angle\ P_1OP_2 = \theta_2 - \theta_1$.

By a theorem from trigonometry (see § 253),

$$\overline{P_1P_2}^2 = \overline{OP_1}^2 + \overline{OP_2}^2 - 2OP_1 \cdot OP_2 \cos \angle\ P_1OP_2.$$

Making the substitutions shown above and taking square roots,

$$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)} \qquad [44]$$

116. The equation of a circle in polar coördinates.—By §115, the equation

$$\sqrt{r^2 + r_1^2 - 2rr_1\cos(\theta - \theta_1)} = a,$$

or

$$r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2 - a^2 = 0$$

represents a circle whose center is (r_1, θ_1) and whose radius is a.

EXERCISES

1. Find the distance between the points (2, 30°) and (3, 60°).

SUGGESTION: $\cos 30° = \frac{1}{2}\sqrt{3} = .8660$.

2. Find the distance between the points (4, 0°) and (4, 60°).

3. Find the equation of the circle with radius 6 which has the point (5, 45°) as its center.

4. Find the equation of the circle which has the point (6, 60°) as its center and which passes through the pole.

117. Polar equation of the general conic.—The following general definition includes as special cases the definitions of all the conic sections.

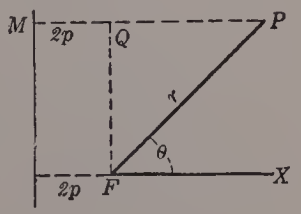
The **conic section**, also called a **conic**, is the locus of a point which moves so that the ratio of its distances from a fixed point and from a fixed line is constant.

The fixed point is called the **focus** and the fixed line is called the **directrix.** The fixed ratio is called the **eccentricity** and is denoted by e.

Let the focus F be the pole of a system of polar coördinates and the line through F perpendicular to the directrix the polar axis. Let $2p$ be the distance from the focus to the directrix.

If P is a point on the curve, then by definition

$$\frac{FP}{MP} = e.$$



But $FP = r$ and $MP = MQ + QP$

$$= 2p + r\cos\theta.$$

Hence $$\frac{r}{2p + r\cos\theta} = e.$$

Solving for r we have

$$r = \frac{2ep}{1 - e\cos\theta} \qquad [45]$$

This is the polar equation of the general conic.

Since $\cos\theta = \cos(-\theta)$ it follows that the conic is symmetrical with respect to the line which passes through its focus and which is perpendicular to the directrix. This line is called the **transverse axis** of the conic. The three conic sections, the parabola, ellipse, and hyperbola, are obtained by putting $e = 1$, $e < 1$, $e > 1$ respectively.

These will now be studied separately. The equations of the "limiting" cases of the conic sections, namely the circle and two straight lines have already been obtained. See §§ 114, 116.

EXERCISES

1. Discuss the conic as to symmetry with respect (a) to the polar axis; (b) to the focus.

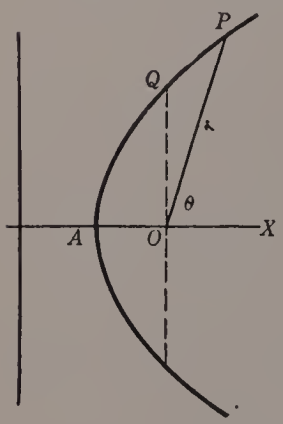
2. Is there any value of θ for which r is zero in the equation of the conic? Does a conic ever pass through its focus? Does it ever meet its directrix?

118. Polar equation of the parabola.—When $e = 1$ the equation of the general conic becomes

$$r = \frac{2p}{1 - \cos\theta},$$

which is the polar equation of the parabola.

The following properties of the parabola are now apparent:



(1) The parabola is symmetrical with respect to the polar axis.

(2) When $\theta = \pi$, $\cos\theta = -1$ and hence $r = p$. That is, the parabola contains a point on its axis which is midway between the focus and the directrix. This point is the **vertex** of the parabola.

(3) When θ decreases from π to $\frac{\pi}{2}$ the point moves along the curve from A to Q in the figure; for $\theta = \frac{\pi}{2}$, $r = 2p$.

(4) When θ decreases from $\frac{\pi}{2}$ to zero the denominator in the equation decreases toward zero and hence the value of r becomes indefinitely great. That is, the parabola extends indefinitely to the right.

(5) As θ increases from π to 2π the point moves along the lower part of the curve and recedes indefinitely toward the right.

EXERCISES

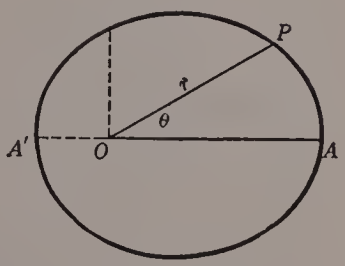
1. What is the latus rectum of the parabola in §118?

2. Find the focal distances of the points on the parabola for which $\theta = \frac{\pi}{3}$, $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{6}$.

3. Find the focal distances of the points on the parabola for which $\theta = \frac{3\pi}{2}$, $\theta = \frac{5\pi}{3}$, $\theta = \frac{7\pi}{4}$, $\theta = \frac{11\pi}{6}$.

119. Polar equation of the ellipse.—By making $e<1$ in the equation of the general conic, we have the equation of the ellipse.

That is, $r = \dfrac{2ep}{1 - e\cos\theta}$.



The following properties are now apparent:

1. When $\theta = 0$, $r = \dfrac{2ep}{1 - e}$.

Since e is positive and less than unity, we have a point A of the ellipse on the principal axis to the right of the focus.

2. When $\theta = \pi$, $r = \dfrac{2ep}{1 + e}$.

This value of r is positive and less than $2p$. Hence there is a point A' of the ellipse on the principal axis between the focus and the directrix. The points A and A' are the **vertices** of the ellipse.

3. Since the value of e is between zero and unity, it follows that the denominator $1 - e\cos\theta$ in the equation of the ellipse cannot equal zero for any value of θ. Moreover, $1 - e\cos\theta$ has its smallest value for $\theta = 0$. Hence the greatest value of r is

$$\frac{2ep}{1 - e}.$$

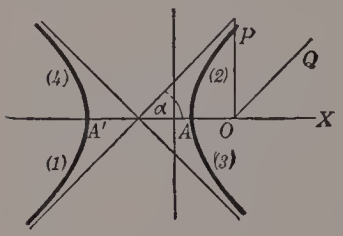
That is, the point A is farther from the focus than any other point on the ellipse.

As θ increases from zero to 2π the point traverses the ellipse completely as indicated in the figure.

EXERCISE

1. Point out the part of the ellipse which is traced by the point, (a) as θ varies from 0 to $\frac{\pi}{2}$, (b) as θ varies from $\frac{\pi}{2}$ to π, (c) as θ varies from π to 2π. Compare the exercises on the hyperbola on page 141.

120. Polar equation of the hyperbola.—By making $e>1$ in the equation of the general conic we have the equation of the hyperbola.



That is, $r = \dfrac{2ep}{1 - e\cos\theta}$.

1. When $\theta = 0$,

$$r = \frac{2ep}{1-e} = -\frac{2ep}{e-1}.$$

Since $e>1$, $\dfrac{2ep}{e-1}$ is positive, and the point $\left(\dfrac{-2ep}{e-1}, 0\right)$ is to the left of the focus. Hence there is a point A' of the hyperbola on the principal axis to the left of the focus.

Since $\dfrac{2ep}{e-1}$ is greater than $2p$, A' is to the left of the directrix.

2. When $\theta = \pi$, $r = \dfrac{2ep}{1+e}$ and hence there is another point, $\left(\dfrac{2ep}{1+e}, \pi\right)$ of the hyperbola which is to the left of the focus. Denote this point by A. Since $\dfrac{e}{1+e}<1$, $\dfrac{2ep}{1+e}<2p$. Therefore A is on the principal axis between the focus and the directrix. The distances from the focus to the points A' and A are $\dfrac{2ep}{e-1}$ and $\dfrac{2ep}{1+e}$.

3. When $\theta = \cos^{-1}\left(\dfrac{1}{e}\right)$ (the angle marked α in the figure) the denominator $1 - e\cos\theta$ equals zero. As θ approaches this value from either side, r grows indefinitely large. As θ increases from zero to $\cos^{-1}\left(\dfrac{1}{e}\right)$, r is negative and the point recedes indefinitely

along the branch marked (1) in the figure. As soon as θ passes the value $\cos^{-1}\left(\frac{1}{e}\right)$, r becomes positive and the point moves in along the branch marked (2). As θ increases from $\cos^{-1}\left(\frac{1}{e}\right)$ to $2\pi - \cos^{-1}\left(\frac{1}{e}\right)$ the point traces the branches marked (2) and (3). When θ passes the value of $2\pi - \cos^{-1}\left(\frac{1}{e}\right)$, r becomes negative and the point moves in along the branch marked (4).

The difference between the parabola and one of the parts of the hyperbola should be noted. There is no point of the hyperbola within the angle XOQ whereas there are points of the parabola within every such angle.

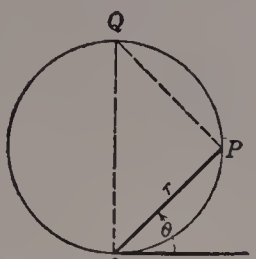
EXERCISE

Point out the part of the hyperbola which is traced by the point, (a) as θ varies from 0 to $\frac{\pi}{2}$, (b) as θ varies from $\frac{\pi}{2}$ to π, (c) as θ varies from π to $\frac{3}{2}\pi$, (d) as θ varies from $\frac{3}{2}\pi$ to 2π. Compare exercise on the ellipse on page 139.

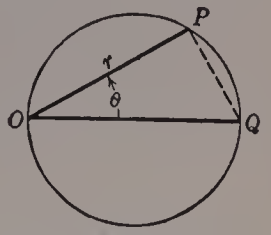
121. Rosettes.—The equations $r = 2a \sin n\theta$ and $r = 2a \cos n\theta$ represent an interesting series of curves for various values of n; these we will now study briefly.

Example 1. Construct the loci of $r = 2a \sin \theta$ and $r = 2a \cos \theta$.

SOLUTION: Construct the two circles shown in the figures making the diameters OQ equal to $2a$. Using the fact that an angle inscribed in a semicircle is a right angle it is at once obvious that the equations of these circles are $r = 2a \sin \theta$ and $r = 2a \cos \theta$.



$r=2a \sin \theta$

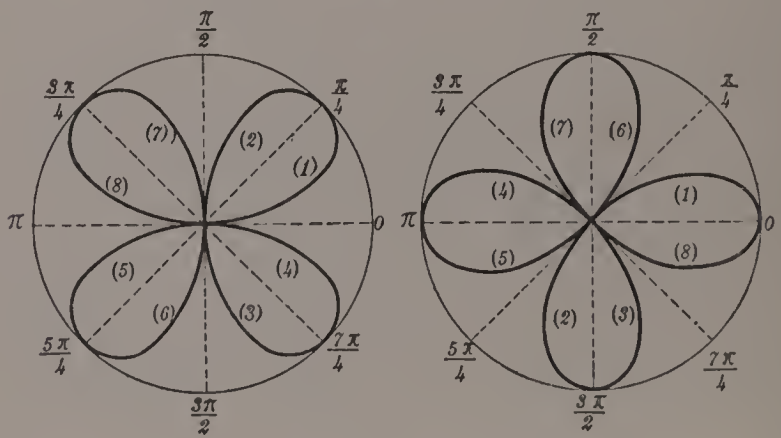


$r=2a \cos \theta$

Example 2. Construct the loci of $r = 2a \sin 2\theta$ and $r = 2a \cos 2\theta$.

SOLUTION: The following tables give corresponding values of θ, 2θ, and r.

	$r = 2a \sin 2\theta$				$r = 2a \cos 2\theta$		
	θ	2θ	r		θ	2θ	r
(1)	0 to $\frac{\pi}{4}$	0 to $\frac{\pi}{2}$	0 to $2a$	(1)	0 to $\frac{\pi}{4}$	0 to $\frac{\pi}{2}$	$2a$ to 0
(2)	$\frac{\pi}{4}$ to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to π	$2a$ to 0	(2)	$\frac{\pi}{4}$ to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to π	0 to $-2a$
(3)	$\frac{\pi}{2}$ to $\frac{3}{4}\pi$	π to $\frac{3}{2}\pi$	0 to $-2a$	(3)	$\frac{\pi}{2}$ to $\frac{3}{4}\pi$	π to $\frac{3}{2}\pi$	$-2a$ to 0
(4)	$\frac{3}{4}\pi$ to π	$\frac{3}{2}\pi$ to 2π	$-2a$ to 0	(4)	$\frac{3}{4}\pi$ to π	$\frac{3}{2}\pi$ to 2π	0 to $2a$
(5)	π to $\frac{5}{4}\pi$	2π to $\frac{5}{2}\pi$	0 to $2a$	(5)	π to $\frac{5}{4}\pi$	2π to $\frac{5}{2}\pi$	$2a$ to 0
(6)	$\frac{5}{4}\pi$ to $\frac{3}{2}\pi$	$\frac{5}{2}\pi$ to 3π	$2a$ to 0	(6)	$\frac{5}{4}\pi$ to $\frac{3}{2}\pi$	$\frac{5}{2}\pi$ to 3π	0 to $-2a$
(7)	$\frac{3}{2}\pi$ to $\frac{7}{4}\pi$	3π to $\frac{7}{2}\pi$	0 to $-2a$	(7)	$\frac{3}{2}\pi$ to $\frac{7}{4}\pi$	3π to $\frac{7}{2}\pi$	$-2a$ to 0
(8)	$\frac{7}{4}\pi$ to 2π	$\frac{7}{2}\pi$ to 4π	$-2a$ to 0	(8)	$\frac{7}{4}\pi$ to 2π	$\frac{7}{2}\pi$ to 4π	0 to $2a$



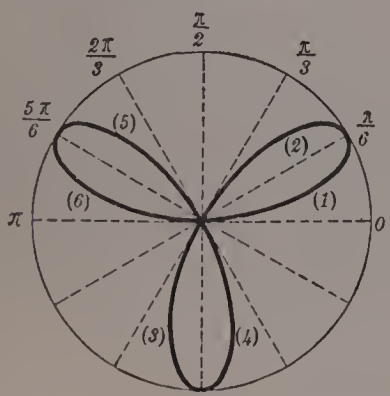
The loci of the equations are shown in the figures, the part of each figure representing the corresponding part of the table being shown by the numerals (1), (2), . . .

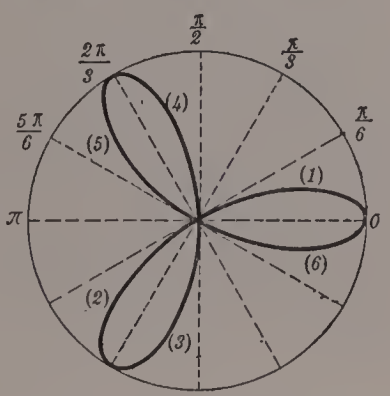
Example 3. Construct the loci of $r = 2a \sin 3\theta$ and $r = 2a \cos 3\theta$.

SOLUTION: The following tables give corresponding values of θ, 3θ, and r.

	θ	3θ	r
(1)	0 to $\frac{1}{6}\pi$	0 to $\frac{\pi}{2}$	0 to $2a$
(2)	$\frac{1}{6}\pi$ to $\frac{1}{3}\pi$	$\frac{\pi}{2}$ to π	$2a$ to 0
(3)	$\frac{1}{3}\pi$ to $\frac{\pi}{2}$	π to $\frac{3}{2}\pi$	0 to $-2a$
(4)	$\frac{\pi}{2}$ to $\frac{2}{3}\pi$	$\frac{3}{2}\pi$ to 2π	$-2a$ to 0
(5)	$\frac{2}{3}\pi$ to $\frac{5}{6}\pi$	2π to $\frac{5}{2}\pi$	0 to $2a$
(6)	$\frac{5}{6}\pi$ to π	$\frac{5}{2}\pi$ to 3π	$2a$ to 0
(7)	π to $\frac{7}{6}\pi$	3π to $\frac{7}{2}\pi$	0 to $-2a$
(8)	$\frac{7}{6}\pi$ to $\frac{4}{3}\pi$	$\frac{7}{2}\pi$ to 4π	$-2a$ to 0

	θ	3θ	r
(1)	0 to $\frac{1}{6}\pi$	0 to $\frac{\pi}{2}$	$2a$ to 0
(2)	$\frac{1}{6}\pi$ to $\frac{1}{3}\pi$	$\frac{\pi}{2}$ to π	0 to $-2a$
(3)	$\frac{1}{3}\pi$ to $\frac{\pi}{2}$	π to $\frac{3}{2}\pi$	$-2a$ to 0
(4)	$\frac{\pi}{2}$ to $\frac{2}{3}\pi$	$\frac{3}{2}\pi$ to 2π	0 to $2a$
(5)	$\frac{2}{3}\pi$ to $\frac{5}{6}\pi$	2π to $\frac{5}{2}\pi$	$2a$ to 0
(6)	$\frac{5}{6}\pi$ to π	$\frac{5}{2}\pi$ to 3π	0 to $-2a$
(7)	π to $\frac{7}{6}\pi$	3π to $\frac{7}{2}\pi$	$-2a$ to 0
(8)	$\frac{7}{6}\pi$ to $\frac{4}{3}\pi$	$\frac{7}{2}\pi$ to 4π	0 to $2a$





Note that in each of these figures, (7) would coincide with (1), (8) would coincide with (2) and so on. Hence the complete locus consists in each case of the three leaves shown in the figures.

122. Solution of Problems.—In the problems following, some of the uses of polar coördinates are illustrated.

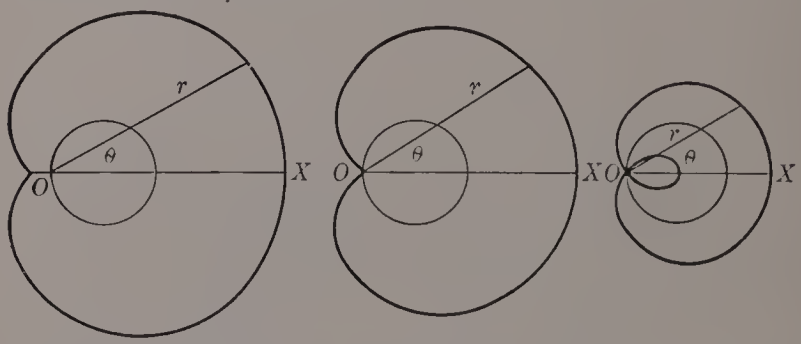
Example 1. A circle with its center on the polar axis passes through the pole; through O (the pole) chords are drawn and each is extended a fixed distance in the positive direction along the radius vector; find the locus of the end-points of the chords thus extended.

SOLUTION: Let $r = 2a\cos\theta$ (see Example 1, page 141) be the equation of the circle and the chords be extended a distance b. Then the equation of the required locus is

$$r = 2a\cos\theta + b.$$

For θ between 0 and $\frac{\pi}{2}$ and between $\frac{3}{2}\pi$ and 2π, $\cos\theta$ is positive and each radius vector is extended by the amount b.

For θ between $\frac{\pi}{2}$ and $\frac{3}{2}\pi$, $\cos\theta$ is negative. For these values of θ, r is found by measuring in the negative direction from O along the radius vector a distance $2a\cos\theta$ and from the point thus reached measuring in the opposite direction a distance b.



The figures represent the cases when $b > 2a$, $b = 2a$, $b < 2a$, respectively. The figure for the case when $b = 2a$ is called the cardioid.

EXERCISES

1. Construct the locus of $r = 2\cos\theta + 4$.
2. Construct the locus of $r = 2\cos\theta + 2$.
3. Construct the locus of $r = 2\cos\theta + 1$.

Example 2. Secants are drawn from the point O meeting a fixed circle in P_1 and P_2; prove that $OP_1 \cdot OP_2$ is a constant.

PROOF: Let O be the pole, (r_1, α) the center of the circle, and a its radius. Then by §116 its equation is

$$r^2 - 2rr_1 \cos(\theta - \alpha) + r_1^2 - a^2 = 0.$$

The two values of r which satisfy this equation are the distances from O to the points in which the secant meets the circle. By § 241 the product of these roots is $OP_1 \cdot OP_2 = r_1^2 - a^2$, which is independent of θ and hence holds for all such secants. (Compare Ex. 6, p. 71).

Example 3. On each focal radius FP' of a conic (extended if necessary), a point P is taken so that $FP = k \cdot FP'$. Find the locus of P.

SOLUTION: Let $r = \dfrac{2ep}{1 - e\cos\theta}$ be the equation of the given conic.

Then $\dfrac{r}{k} = \dfrac{2ep}{1 - e\cos\theta}$, or $r = \dfrac{2epk}{1 - e\cos\theta}$, is the required equation.

The locus is a conic having the same focus and eccentricity as the given conic, the distance from the focus to the directrix being multiplied by k (compare page 74). Note that k may be either greater or less than unity.

EXERCISES

1. In Example 2 may the point O be either outside or inside the circle? What theorems in plane geometry are proved in this example?

2. If in Example 3 the conic is an ellipse or a hyperbola how are the semiaxes a and b affected? How is distance between the foci affected? the distance between the directrices?

3. Note that in $r = 2a \sin 2\theta$ and $r = 2a \cos 2\theta$ (see page 142) the rosettes consist of four leaves, while in $r = 2a \sin \theta$ and $r = 2a \sin 3\theta$, as also in $r = \cos\theta$ and $r = 2a \cos 3\theta$, the rosettes consist of one and three leaves respectively (see pages 141 and 143). Try to establish the following law:

If n is odd, $r = 2a \sin n\theta$ and $r = 2a \cos n\theta$ represent figures consisting of n leaves or petals.

If n is even, $r = 2a \sin n\theta$ and $r = 2a \cos n\theta$ represent figures consisting of $2n$ leaves.

PROBLEMS ON CHAPTER IX

1. Through a fixed point O on a fixed circle chords OP' are drawn, and on each of these (extended if necessary) a point P is taken so that $OP = k \cdot OP'$; find the locus of P.

SUGGESTION: Place the circle so that O is the pole and its center is on the polar axis.

2. Find the polar equation of a line perpendicular to the radius vector at the point (a, α). Compare §32.

3. In a conic the lengths of the two focal radii along the same focal chord are r_1 and r_2; prove that $\frac{1}{r_1} + \frac{1}{r_2}$ is a constant (the same for all focal chords).

4. Prove that in a parabola the length of the focal chord which makes an angle of 30° with the polar axis is four times the length of the latus rectum.

5. In a hyperbola the focal chord FP parallel to an asymptote meets the curve in P; show that the latus rectum is four times FP.

6. Find the locus of the middle points of the focal radii of a conic.

7. Find the locus of the point P which divides the focal radii of a conic in a fixed ratio. Consider the cases when the division is internal and when it is external.

In 8–11 transform the equation into polar coördinates.

8. $y^2 = \dfrac{x^3}{2p - x}$ (see §168).

9. $x^2y^2 = (y + a)^2(b^2 - y^2)$ (see §169).

10. $x^2y = 4k^2(2p - y)$ (see §170).

11. $x^3 + y^3 = 3axy$ (see §176).

In 12–14 transform into rectangular coördinates.

12. $r = 2 \cos \theta + 4$ (see §122, Ex. 1).

13. $r = 2a \sin 2\theta$ (see §121, Ex. 2).

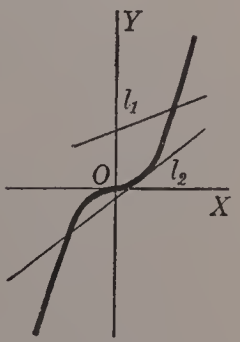
14. $r = 2a \cos 2\theta$.

15. Lines OQ through a fixed point O are drawn meeting a fixed line in Q; find the locus of P if $OP \cdot PQ$ is a constant.

CHAPTER X

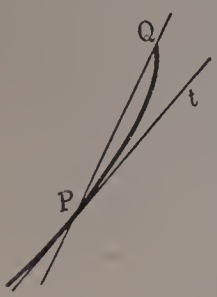
TANGENTS TO CONICS

The definition of tangent to a curve which we have used up to this point, namely a straight line which meets the curve in only one point, is adequate for conic sections except the parabola and hyperbola, which may be met in a single point by a straight line not a tangent. It serves also for the parabola when we except all lines parallel to its axis, and for the hyperbola when we except lines parallel to the asymptotes.



In general, however, this definition is not adequate. Not only may a line meet a curve in only one point and not be a tangent to it, but it may also meet the curve in two (or more) points and still be a tangent to it. Thus, in the figure, line l_1 meets the curve $y = x^3$ in only one point and is not a tangent, while l_2 meets it in two points and is a tangent.

123. General definition of tangent to a curve.—Because of the facts just noted it is necessary to formulate a definition of tangent to a curve different from that which we have used up to this point. The following definition of tangent applies to all curves which have tangents:



If a line meets a curve in two points P *and* Q *and if the point* Q *is made to move along the curve so as to approach* P *as a limit, and if as this takes place the line* PQ *approaches a fixed line* t *as a limit, then* t *is tangent to the curve at the point* P.

124. Equation of tangent to a conic at a given point.—Let the equation of the conic be

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0 \tag{1}$$

and let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on it.

By §25 the slope of the line through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\frac{y_2 - y_1}{x_2 - x_1}. \tag{2}$$

Since P and Q lie on the conic,

$$Ax_1^2 + By_1^2 + 2Dx_1 + 2Ey_1 + C = 0 \tag{3}$$

$$Ax_2^2 + By_2^2 + 2Dx_2 + 2Ey_2 + C = 0. \tag{4}$$

Subtracting (3) from (4), dividing by $x_2 - x_1$, and solving for $\frac{y_2 - y_1}{x_2 - x}$, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{2D + A(x_2 + x_1)}{2E + B(y_2 + y_1)}.$$

Letting Q approach P as a limit we have

$$\text{limit } \frac{y_2 - y_1}{x_2 - x_1} = \text{limit } \left(-\frac{2D + A(x_2 + x_1)}{2E + B(y_2 + y_1)}\right) = -\frac{D + Ax_1}{E + By_1}.$$

Hence the slope of the tangent at $P(x_1, y_1)$ is $-\frac{D + Ax_1}{E + By_1}$ and its equation is

$$y - y_1 = -\frac{D + Ax_1}{E + By_1}(x - x_1),$$

which reduces to

$$Ax_1x + By_1y + Dx + Ey - Dx_1 - Ey_1 - Ax_1^2 - By_1^2 = 0. \tag{5}$$

Adding $Dx_1 + Ey_1$ to (5) and also subtracting the same quantities, we have

$$Ax_1x + By_1y + D(x + x) + E(y + y_1) - (Ax_1^2 + By_1^2 + 2Dx_1 + 2Ey_1) = 0.$$

But from (3), $Ax_1^2 + By_1^2 + 2Dx_1 + 2Ey_1 = -C$.

Hence, $\boldsymbol{Ax_1x + By_1y + D(x + x_1) + E(y + y_1) + C = 0}$ [46]

which is the equation of the tangent to (1) at the point $P(x_1, y_1)$.

125. Tangents to the separate conics.—Equation [46] on page 148 contains as special cases the equations of the tangents to the circle, parabola, ellipse, and hyperbola which were found in chapters IV–VII.

Suppose that in this equation $A = B = 1$, $D = E = 0$, and $C = -r^2$.

For these values of the coefficients, equation [46] reduces to

$$x_1x + y_1y = r^2. \qquad \text{(See §51)}$$

If $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $D = E = 0$, and $C = -1$, then [46] reduces to

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1. \qquad \text{(See §76)}$$

If $A = \frac{1}{a^2}$, $B = -\frac{1}{b^2}$, $D = E = 0$, and $C = -1$, then [46] reduces to

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1. \qquad \text{(See §90)}$$

If $A = 0$, $B = 1$, $D = -2p$, and $E = C = 0$, then [46] reduces to $y_1y - 2p(x + x_1) = 0$, or $y_1y = 2p(x + x_1)$. (See §61)

Finally, if $A = 1$, $B = 0$, $E = -2p$, and $D = C = 0$, then [46] reduces to

$$x_1x - 2p(y + y_1) = 0, \quad \text{or} \quad x_1x = 2p(y + y_1).$$

EXERCISES

Find the equations of the tangents to the following conics at the points indicated.

1. $x^2 + y^2 - 4x + 5y + 10 = 0$ at the points where $x = 2$.

SUGGESTION: Substitute $x = 2$ and find the corresponding values of y.

2. $x^2 + 4y^2 - 10x - 12y + 8 = 0$ at the points where $x = 4$.
3. $6x^2 - y^2 + 2x - 16y + 15 = 0$ at the points where $x = 10$.
4. $9y^2 - 10x + 4y - 30 = 0$ at the points where $x = 5$.

126. A modification of the definition of tangent.—In finding the equations of tangents we have already used, by implication, a definition which is a modification of the general definition given in this chapter. We will illustrate with an example.

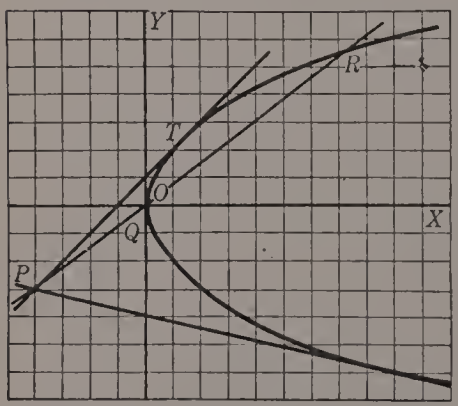
Example. Find the equations of the tangents to the parabola $y^2 = 4x$ which pass through the point $(-4, -3)$.

SOLUTION: Solving simultaneously

$$y^2 = 4x \text{ and } y + 3 = m(x + 4),$$

$$y = \frac{2 \pm \sqrt{4 - m(16m - 12)}}{m}.$$

The expression under the radical sign is zero for $m = 1$ and $m = -\frac{1}{4}$, and these values make the line tangent to the curve.



For a general value of m let the line meet the curve in Q and R. Then let the line swing about the point P so that m approaches unity. Then the points Q and R both move along the curve until they coincide at T, when the line becomes tangent. In this case the following definition of tangent is used.

If the points Q *and* R *both approach the point* T *along the curve, then the line* QR *approaches the tangent at* T.

This definition is the basis for the following:

If the equation of a straight line is solved simultaneously with the equation of a curve and if two (or more) roots coincide, then the line is a tangent to the curve.

The definition just given is equivalent to that given on page 147 for all cases where the curve has a tangent, which is the case for all curves except some special cases considered in higher mathematics. In such special cases the secant PQ used on page 147 does not approach a definite limit as Q approaches P. However, the method used in the example above might easily yield a definite result in such a case, but the line found would not be a tangent.

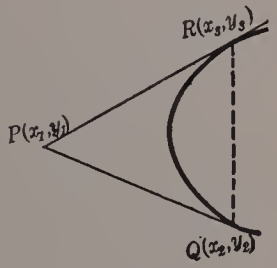
CHAPTER XI

POLES AND POLARS

We have seen that through any point P there are two tangents, or one tangent, or no tangent to a conic. If there are two tangents, the point is **outside the conic,** if there is one tangent, the point is **on the conic,** and if there is no tangent through the point, it is **inside the conic.**

127. Chord of contact.—Let $P(x_1, y_1)$ be any point from which there are two tangents to the conic

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0.$$



Let the points of contact of the tangents be $Q(x_2, y_2)$ and $R(x_3, y_3)$.

Then the equations of the tangents are (see §124)

$$Ax_2x + By_2y + D(x_2 + x) + E(y_2 + y) + C = 0 \qquad (1)$$

$$Ax_3x + By_3y + D(x_3 + x) + E(y_3 + y) + C = 0. \qquad (2)$$

Since these two lines both pass through the point $P(x_1, y_1)$,

$$Ax_2x_1 + By_2y_1 + D(x_2 + x_1) + E(y_2 + y_1) + C = 0 \qquad (3)$$

and $$Ax_3x_1 + By_3y_1 + D(x_3 + x_1) + E(y_3 + y_1) + C = 0. \qquad (4)$$

Now consider the equation

$$Ax_1x + By_1y + D(x_1 + x) + E(y_1 + y) + C = 0. \qquad (5)$$

From equations (3) and (4) we have that equation (5) is satisfied by x_2, y_2 and by x_3, y_3. But (5) is a first degree equation in x and y and hence represents a straight line. Therefore (5) is the equation of the straight line which passes through the points of contact of the tangents from $P(x_1, y_1)$. This line is the **chord of contact** of the tangents PQ and PR.

128. Pole and polar.—If $P(x_1, y_1)$ is any point in the plane of the conic

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0, \tag{1}$$

then the line

$$Ax_1x + By_1y + D(x_1 + x) + E(y_1 + y) + C = 0 \tag{2}$$

is called the **polar** of the point $P(x_1,y_1)$ with respect to the conic, and $P(x_1,y_1)$ is called the **pole** of this line with respect to the conic.

We have already seen that

(*1*) *in case* P *lies on the conic its polar is the tangent to the conic at that point* (see §124);

(*2*) *in case* P *lies outside the conic, its polar is the chord of contact of the tangents from* P *to the conic* (see §127).

We shall now show that

(*3*) *in case* P *lies within the conic its polar is a line which fails to meet the conic.*

If possible, let the polar of such a point, $P(x_1, y_1)$, meet the conic in a point $Q(x_2, y_2)$.

Then $Ax_1x + By_1y + D(x_1 + x) + E(y_1 + y) + C = 0$ is satisfied by x_2, y_2.

That is, $Ax_1x_2 + By_1y_2 + D(x_1 + x_2) + E(y_1 + y_2) + C = 0$ and hence the tangent at $Q(x_2, y_2)$,

$$Ax_2x + By_2y + D(x_2 + x) + E(y_2 + y) + C = 0,$$

passes through the point $P(x_1, y_1)$. But this contradicts the hypothesis that P lies within the conic and that hence no tangent can pass through P.

EXERCISES

1. Find the equation of the polar of the point $(-5, -4)$ with respect to the conic $x^2 - 4y^2 - 2x + 16y - 51 = 0$.

2. Find the points in which the polar found in Ex. 1 meets the given conic.

3. Find the tangents of the conic in Ex. 1 which pass through the point $(-5,-4)$.

129. Finding the pole of a given line.—When a line and a conic are given, the pole of the line with respect to the conic may be found as follows:

Let the equations of the conic and the line be

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0 \tag{1}$$

and

$$ax + by + c = 0. \tag{2}$$

Suppose the pole of (2) with respect to (1) is (x_1, y_1).

Then the equation of the polar is

$$Ax_1x + By_1y + D(x_1 + x) + E(y_1 + y) + C = 0 \tag{3}$$

in which A, B, D, E, C are known, and x_1 and y_1 are to be determined.

Equation (3) may be reduced to

$$(Ax_1 + D)x + (By_1 + E)y + Dx_1 + Ey_1 + C = 0. \tag{4}$$

But (4) is equivalent to (2).

Hence $\dfrac{b}{a} = \dfrac{By_1 + E}{Ax_1 + D}$ (5), $\qquad \dfrac{c}{a} = \dfrac{Dx_1 + Ey_1 + C}{Ax_1 + D}$. (6)

These two equations may be solved for the two unknowns x_1 and y_1 and hence the pole (x_1, y_1) of the given line may be found.

If, however, the given line is a diameter of the conic (see Chapter XII), equations (5) and (6) will be found to be contradictory and hence a diameter of a conic has no pole.

130. Every point except one has a polar.—If in the equation

$$Ax_1x + By_1y + D(x_1 + x) + E(y_1 + y) + C = 0 \tag{1}$$

we substitute $x_1 = -\dfrac{D}{A}$, $y_1 = -\dfrac{E}{B}$, then the coefficients of both x and y become zero. Hence the point $\left(-\dfrac{D}{A}, -\dfrac{E}{B}\right)$ has no polar. Evidently every other point has a polar since the substitution of any other pair of numbers in (1) gives a definite line.

It should be noticed that in case $A = 0$ or $B = 0$, there is no such point as $\left(-\dfrac{D}{A}, -\dfrac{E}{B}\right)$. That is, in the case of the parabola every point has a polar. In the case of the other conics (the central conics) the center has no polar.

131. Special relation between pole and polar.—Let $P_1(x_1,y_1)$ be any point, let

$$Ax_1x + By_1y + D(x_1 + x) + E(y_1 + y) + C = 0 \qquad (1)$$

be its polar, p_1, and let $P_2(x_2, y_2)$ be any point on p_1.

Then $Ax_1x_2 + By_1y_2 + D(x_1 + x_2) + E(y_1 + y_2) + C = 0.$ (2)

But the polar p_2 of P_2 is

$$Ax_2x + By_2y + D(x_2 + x) + E(y_2 + y) + C = 0. \qquad (3)$$

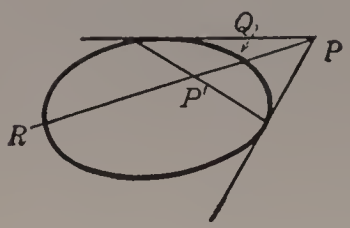
From (2) it follows that p_2 passes through P_1.

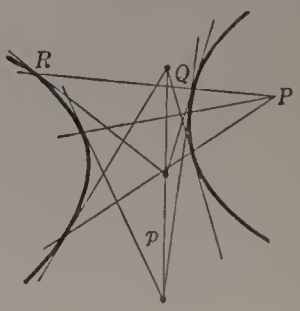
Hence if P_2 is a point on the polar of P_1 then the polar of P_2 passes through P_1.

In a similar manner we may show that if p_3 is a line through P_1, then the pole of p_3 lies on the polar of P_1. The two propositions just obtained may be stated as follows:

The poles of all lines through a point lie on the polar of that point, and the polars of all points on a line pass through the pole of that line.

132. Another definition of pole and polar.—It follows from the statements just made that if through a point P (whether inside or outside the conic) a secant is drawn meeting the conic in Q and R, then the tangents at Q and R will meet on the polar of P. Hence the points in which all such pairs of tangents meet lie on a straight line.





A definition of pole and polar is sometimes based on this fact, namely:

The polar of a point P *is the straight line in which tangents meet if they are drawn at the points in which secants through* P *meet the conic.*

133. Another general property of poles and polars.—We will now prove a general theorem upon which still another definition of pole and polar may be based.

THEOREM: *If* p *is the polar of* P *with respect to a conic* C, *and if a line through* P *meets the conic in* Q *and* R *and the line* p *in* P′, *then*

$$PQ/PR = -P'Q/P'R.$$

PROOF: If these points are $P(x_1, y_1)$, $P'(x_4, y_4)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, then we need to prove that

$$\frac{x_1 - x_2}{x_1 - x_3} = -\frac{x_4 - x_2}{x_4 - x_3} \quad (1) \quad \text{and} \quad \frac{y_1 - y_2}{y_1 - y_3} = -\frac{y_4 - y_2}{y_4 - y_3}. \quad (2)$$

Translate the axes (§101) so that P is at the origin.

Then (1) and (2) reduce to

$$2x_2x_3 = x_4(x_2 + x_3) \quad (3) \qquad \text{and} \quad 2y_2y_3 = y_4(y_2 + y_3). \quad (4)$$

If the equation of the conic after translating the axis is

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0, \quad (5)$$

then the polar of the origin with respect to (5) is

$$Dx + Ey + C = 0. \quad (6)$$

Let $y = mx$ be the line passing through the points $PQP'R$.

Substituting $y = mx$ in (5) we have

$$(A + Bm^2)x^2 + 2(D + Em)x + C = 0.$$

The roots of this equation are x_2 and x_3.

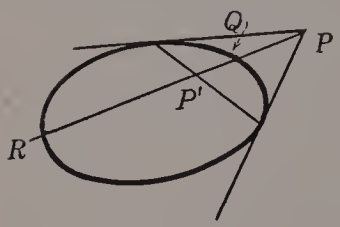
Hence $2x_2 \cdot x_3 = \dfrac{2C}{A + Bm^2}$ and $x_2 + x_3 = -\dfrac{2(D + Em)}{A + Bm^2}$ (see §241).

From (6) and $y = mx$ we find $x_4 = -\dfrac{C}{D + Em}$.

Hence $x_4(x_2 + x_3) = \dfrac{-2(D + Em)}{A + Bm^2} \cdot \dfrac{-C}{D + Em} = \dfrac{2C}{A + Bm^2}$.

Therefore $2x_2x_3 = x_4(x_2 + x_3)$. In the same manner we may show that $2y_2y_3 = y_4(y_2 + y_3)$, and the theorem is proved.

134. Harmonic division of secant by pole and polar.—If the collinear points $PP'QR$ are so related that $PQ/PR = -P'Q/P'R$, then QR is said to be divided harmonically by P and P'. The following definition may therefore be based on §133.



If the secants QR *of a conic through a fixed point* P *are divided harmonically by* PP′ *then the locus of* P′ *is the polar of* P.

EXERCISES

Find the polar of

1. (8, 5) with respect to $x^2 + y^2 = 16$.
2. $(-7, 4)$ with respect to $y^2 = 4px$.
3. (10, 4) with respect to $25x^2 + 49y^2 = 1225$.
4. (3, 2) with respect to $25x^2 - 49y^2 = 1225$.
5. (5, 2) with respect to $x^2 + y^2 - 8x + 4y - 20 = 0$.
6. (2, 3) with respect to $8x^2 + 24y^2 - 15x + 18y - 10 = 0$.
7. (8, 2) with respect to $x^2 - 8y^2 - 4x + 6y - 60 = 0$.

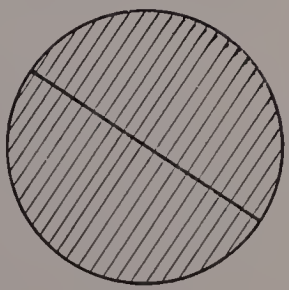
Find the pole of

8. $x - 3y + 2 = 0$ with respect to $x^2 + y^2 = 9$.
9. $2x + y - 4 = 0$ with respect to $y^2 = 10x$.
10. $x + 4y + 10 = 0$ with respect to $x^2 = 12y$.
11. $x - y + 6 = 0$ with respect to $9x^2 + 4y^2 = 36$.
12. $x + 2y - 8 = 0$ with respect to $9x^2 - 4y^2 = 36$.
13. $2x - 3y - 4 = 0$ with respect to $3x^2 + 8y^2 - 9x + 8 = 0$.
14. $3x - 7y - 8 = 0$ with respect to $4x^2 - 12y^2 + 13y - 16 = 0$.
15. $5x - 4y + 4 = 0$ with respect to $2x^2 - 7y^2 + 12y - 8 = 0$.
16. Show that for any conic section the polar of a focus is a directrix.
17. If the point P approaches the center of a circle, describe the change in its polar.
18. Describe the change in the polar of P (a) with respect to an ellipse, (b) with respect to a hyperbola, as P approaches the center.

CHAPTER XII

DIAMETERS

It is obvious that in a circle the middle points of a system of parallel chords lie on a diameter of the circle. We will now consider the locus of the middle points of parallel chords of any conic.



135. Middle points of a system of parallel chords.—

Let the line

$$y = mx + b \qquad (1)$$

meet the conic

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0 \qquad (2)$$

in the points $Q(x_1, y_1)$, $R(x_2, y_2)$.

Then $\bar{x} = \dfrac{x_1 + x_2}{2}$, $\bar{y} = \dfrac{y_1 + y_2}{2}$, where $(\bar{x}, \bar{y})$ is the middle point of the chord QR.

Substituting (1) in (2) and rearranging terms,

$$x^2(A + Bm^2) + 2(Bbm + Em + D)x + Bb^2 + 2Eb + C = 0.$$

Then by §241

$$\bar{x} = \frac{x_1 + x_2}{2} = -\frac{Bbm + Em + D}{A + Bm^2}. \qquad (3)$$

Since $(\bar{x}, \bar{y})$ lies on $y = mx + b$, $b = \bar{y} - m\bar{x}$.

Substituting this value of b in (3),

$$\bar{x} = -\frac{Bm(\bar{y} - m\bar{x}) + Em + D}{A + Bm^2}.$$

Suppressing dashes and simplifying,

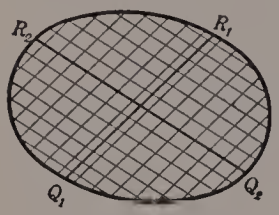
$$Ax + Bmy + Em + D = 0. \qquad (4)$$

Equation (4) holds for the middle points of all chords whose slope is m.

Hence we have:

The middle points of a system of parallel chords lie on a straight line.

136. Diameters.—The straight line bisecting a system of parallel chords of a conic is called a **diameter** of the conic. It follows from §135 that any system of parallel chords determines a diameter of the conic.



If the slope of the chords of a given system is m, then from equation (4), page 157, the slope of the diameter bisecting the chords is $-\frac{A}{Bm}$.

137. Conjugate diameters.—If a diameter meets a conic in two points, then one of the chords which it bisects passes through the middle point of the diameter. We will now prove the following proposition:

If the chord Q_1R_1 *(see the figure in §136) bisects the diameter* Q_2R_2 *of the system of chords parallel to* Q_1R_1, *then* Q_1R_1 *is the diameter of the system of chords parallel to* Q_2R_2.

PROOF: If the slope of Q_1R_1 is m, then by §136 the slope of Q_2R_2 is $-\frac{A}{Bm}$. Hence if the slope of Q_2R_2 is $-\frac{A}{Bm}$, then the slope of the diameter which bisects the chords parallel to Q_2R_2 is

$$\frac{A}{-B\left(-\frac{A}{Bm}\right)} = m.$$

That is, the slope of this diameter is the same as the slope of Q_1R_1. But Q_1R_1 and the diameter both bisect Q_2R_2. Hence Q_1R_1 is the diameter which bisects the set of chords parallel to Q_2R_2.

We may therefore adopt the following definition:

If each of two diameters bisects the chords parallel to the other, they are called **conjugate diameters.**

The word diameter is really used with two different meanings, namely, to indicate the complete line and also the segment between the points in which it cuts the conic.

138. The polars of points on a diameter.—The relation between diameters and poles and polars is given by the following proposition:

If a point lies on a diameter bisecting a set of chords of slope m, *then the polar of this point has a slope* m.

PROOF: Let (x_1, y_1) be the given point, and $Ax^2 + By^2 + 2Dx + 2Ey + C = 0$ the given conic. Since the point (x_1, y_1) lies on the diameter bisecting the chords with slope m, we have by §135

$$Bmy_1 + Ax_1 + Em + D = 0.$$

Hence
$$m = -\frac{Ax_1 + D}{By_1 + E}. \tag{1}$$

The polar of (x_1, y_1) is by §128

$$Ax_1x + By_1y + D(x_1 + x) + E(y_1 + y) + C = 0. \tag{2}$$

The slope of this line is $-\dfrac{Ax_1 + D}{By_1 + E}$, which by (1) is equal to m. Hence the line (2) has a slope m, which was to be proved.

139. Tangents at points where diameter meets conic.—The following is an important special case of the general theorem of §138.

If a diameter meets the conic in a point P, *then the tangent at* P *is parallel to the system of chords bisected by the diameter.*

To prove this proposition we only have to notice that the polar of a point on a conic is the tangent to the conic at that point.

The following is now obvious:

The tangents at the extremities of a chord meet on the diameter bisecting this chord.

EXERCISES

1. Find the equation of the diameter of $x^2 + 3y^2 - 4x + 6y - 10 = 0$ which bisects the chords whose slopes are $\frac{1}{2}$.

2. Find the equation of the diameter of $3x^2 - 6y^2 - 9x - 4y + 10 = 0$ which has a slope of $-\frac{3}{4}$.

140. Diameters of the circle.—If in the equation

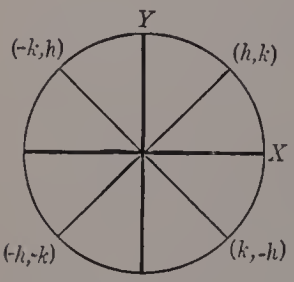
$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0$$

we put $A = B = 1$, $D = E = 0$, and $C = -r^2$, we obtain

$$x^2 + y^2 = r^2,$$

which is the equation of the circle with the center at the origin.

Then the equation of the diameter bisecting the chords with slope m becomes $y = -\frac{x}{m}$.



Hence in the circle a pair of conjugate diameters are perpendicular to each other.

If the point (h, k) is an extremity of a diameter, then the other extremity is $(-h, -k)$ and the extremities of the conjugate diameter are $(-k, h)$, $(k, -h)$.

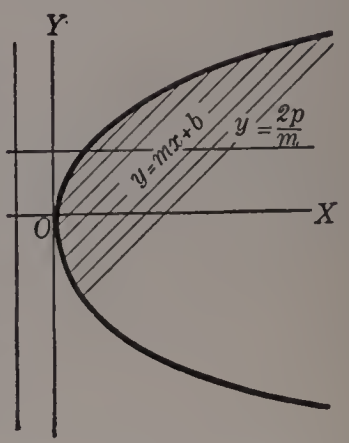
141. Diameters of parabola.—If in the equation $Ax^2 + By^2 + 2Dx + 2Ey + C = 0$ of the conic we put $A = 0$, $B = 1$, $D = -2p$, $E = C = 0$, we obtain

$$y^2 = 4px,$$

which is the equation of the parabola.

Then the equation of the diameter which bisects the chords with slope m is

$$y = \frac{2p}{m},$$



which is a line parallel to the x-axis. Hence every diameter of a parabola is parallel to the axis, and conversely, every line parallel to the axis is a diameter. Since every diameter meets the parabola in only one point, it follows that a parabola has no pair of conjugate diameters.

From the equation $y = \frac{2p}{m}$ it is apparent that when m is large the diameter is near the axis of the parabola, and when m is small, the diameter is farther from the axis.

Evidently the line $y = 0$ bisects all chords at right angles to it, and is therefore a diameter. However, this line cannot be expressed by the equation $y = \frac{2p}{m}$ since a chord perpendicular to the x-axis does not have a slope in the proper sense of the word.

EXERCISES

1. In the parabola $y^2 = 8x$ a diameter meets the curve in the point where $y = 6$; find the slope of the chords bisected by this diameter.

2. In the parabola $y^2 = 10x$ a diameter bisects the chords whose slope is $\frac{1}{4}$; find the equation of the tangent at the point in which this diameter meets the curve.

3. In the conic $2x^2 - 7y^2 + 10y + 8 = 0$ a diameter has a slope $\frac{3}{5}$; find the equations of this diameter and of its conjugate.

4. Find the equation of that diameter of $3x^2 + 9y^2 - 16x + 10 = 0$ which passes through the point $(3, 7)$.

5. Show that the diameter of $Ax^2 + By^2 + 2Dx + 2Ey + C = 0$ which has a given slope does not change in position when C changes.

6. In the conic $x^2 - 8y^2 + 2Dx - 10y + 12 = 0$ find the value of D for which the diameter through the point $(2, 4)$ has a slope $\frac{5}{6}$.

7. In the conic $x^2 - 4y^2 - 4x + 8y - 36 = 0$ a diameter bisects the chords whose slope is $\frac{5}{8}$; find its equation.

8. Find the equations of the tangents to the conic in Ex. 7 at the points in which the diameter meets the conic.

9. Find the value of E in $7x^2 + 2y^2 + 12x + 2Ey + C = 0$ for which a diameter with slope $-\frac{3}{4}$ passes through the point $(-2, 5)$.

10. The tangent to the parabola $y^2 = 12x$ at the point $(3, 6)$ is a diameter of the conic $Ax^2 - 10y^2 + 8x - 6y + C = 0$; find the value of A.

142. Diameters of the ellipse.—If in the equation of the conic we put $A = b^2$, $B = a^2$, $D = E = 0$, and $C = -a^2b^2$, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the equation of the ellipse in its simplest form.

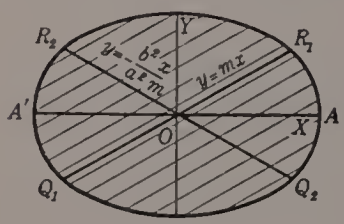
Then the equation of the diameter bisecting the chords with slope m becomes

$$y = -\frac{b^2x}{a^2m}.$$

Hence every diameter of the ellipse passes through the center.

From the slopes m and $-\frac{b^2}{a^2m}$ of a pair of conjugate diameters, it follows that they cannot be at right angles to each other unless $a = b$, in which case the ellipse becomes a circle.

The two axes of the ellipse, however, form a pair of conjugate diameters and are at right angles to each other. One of these diameters, the y-axis, cannot be expressed in the form $y = mx + b$, since a line at right angles to the x-axis does not have a slope.



If one of the slopes m and $-\frac{b^2}{a^2m}$ is positive, the other is negative. Hence, one of a pair of conjugate diameters makes an acute angle with the x-axis and the other makes an obtuse angle with it.

EXERCISES

1. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ a diameter passes through the point (3, 4) and its conjugate diameter passes through the point $(-25, 9)$; find the ratio $b : a$.

2. Describe the change of the angle between two conjugate diameters of an ellipse as the slope of one of them approaches zero.

143. Extremities of a diameter in terms of those of its conjugate diameter.—If (h,k) is an extremity of a given diameter of an ellipse, then $(-h, -k)$ is its other extremity. If m is the slope of this diameter, then $m = k/h$. Substituting this value of m in $y = -\dfrac{b^2x}{a^2m}$ we have

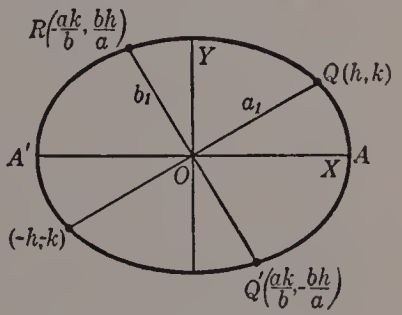
$$a^2ky = -hb^2x, \tag{1}$$

which is the equation of the diameter that is conjugate to the given diameter. Solving (1) and $b^2x^2 + a^2y^2 = a^2b^2$ simultaneously and noting that since (h, k) is on the ellipse, $b^2h^2 + a^2k^2 = a^2b^2$, we have

$$x = ak/b,\ y = -bh/a;\ x = -ak/b,\ y = bh/a.$$

These are the extremities of the diameter conjugate to the given diameter.

144. Lengths of conjugate diameters.—Let $Q\,(h, k)$ and $R\left(-\dfrac{ak}{b}, \dfrac{bh}{a}\right)$ be extremities of a pair of conjugate diameters of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Denote the lengths of the semidiameters OQ and OR by a_1 and b_1 respectively.



Then $$a_1^2 = h^2 + k^2 \quad \text{and} \quad b_1^2 = \frac{a^2k^2}{b^2} + \frac{b^2h^2}{a^2}.$$

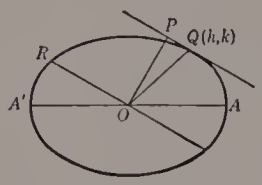
Hence $$a_1^2 + b_1^2 = h^2 + k^2 + \frac{a^2k^2}{b^2} + \frac{b^2h^2}{a^2}.$$

Noting that (h, k) is on the ellipse, the second member of this equation reduces to $a^2 + b^2$.

Hence $$a_1^2 + b_1^2 = a^2 + b^2.$$

That is, the sum of the squares of two conjugate semidiameters is constant.

145. The angle formed by conjugate diameters.—Let OQ and OR be any two conjugate semidiameters of the ellipse. At $Q(h, k)$ draw a tangent to the ellipse, and let OP be perpendicular to it.



By §139, QP is parallel to OR, and hence $\sin QOR = \sin OQP$.

The equation of QP is by §76

$$b^2hx + a^2ky = a^2b^2, \tag{1}$$

and by §41 the distance from the origin to this tangent is

$$OP = \frac{a^2b^2}{\sqrt{b^4h^2 + a^4k^2}}.$$

Hence $$\sin OQP = \frac{OP}{OQ} = \frac{a^2b^2}{\sqrt{b^4h^2 + a^4k^2}} \cdot \frac{1}{OQ}.$$

But $$OR = \sqrt{\frac{a^2k^2}{b^2} + \frac{b^2h^2}{a^2}} = \frac{1}{ab}\sqrt{a^4k^2 + b^4h^2}.$$

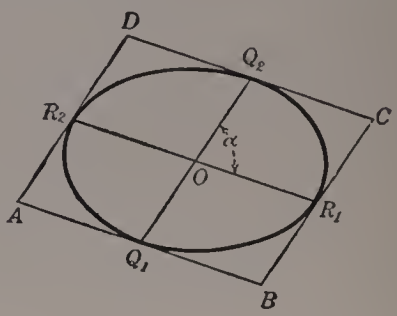
Denote OQ and OR by a_1 and b_1 respectively.

Then $$\sqrt{a^4k^2 + b^4h^2} = abb_1.$$

Hence $$\sin OQP = \frac{a^2b^2}{abb_1} \cdot \frac{1}{a_1} = \frac{ab}{a_1b_1}.$$

That is, *the sine of the angle formed by two conjugate diameters whose half-lengths are* a_1 *and* b_1 *is* ab/a_1b_1.

146. Parallelogram on conjugate diameters.—Let Q_1Q_2 and R_1R_2 be a pair of conjugate diameters of an ellipse. Construct tangents at Q_1, Q_2, R_1, and R_2 forming the parallelogram $ABCD$.



Then the area of the parallelogram OR_1CQ_2 is

$$OR_1 \cdot OQ_2 \sin \alpha = a_1b_1 \sin \alpha$$

$$= a_1b_1 \cdot \frac{ab}{a_1b_1} = ab.$$

Hence the area of $ABCD$ is $4\,ab$.

That is, the area of such a parallelogram is constant.

EXERCISES

1. In the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ the length of a diameter is 10; find the length of the conjugate diameter.

2. Find the angle formed by the pair of conjugate diameters in Ex. 1.

3. Find the equations of the tangents at the extremities of the diameters in Ex. 1.

4. Discuss the problem of finding the pole of a diameter for any conic.

SUGGESTION: Consider the pole of a chord which moves parallel to itself so as to approach a diameter.

5. In the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ the sine of the angles formed by a pair of conjugate diameters is .9; find the lengths of these diameters.

6. In the ellipse $\frac{x^2}{64} + \frac{y^2}{16} = 1$ the length of a diameter is 12; find the sine of the angle which this diameter makes with its conjugate diameter.

7. In the ellipse of Ex. 6 the slope of a diameter is $-\frac{1}{2}$; find the length of its conjugate diameter.

8. In a pair of conjugate diameters of lengths a_1 and b_1 which do not coincide with the axes of the ellipse show that a_1 and b_1 are both greater than b and less than a.

9. In an ellipse is there more than one pair of conjugate diameters which make a given angle? how many?

10. Show that the tangents at the extremities of each of a set of parallel chords meet on a line. How is this line related to the chords?

11. How are the polars of the points on a diameter related? As a point moves from the center of an ellipse along a diameter how does the corresponding polar move?

12. In the ellipse $4x^2 + 9y^2 + 10x - 4y + 2 = 0$ find a relation between the slopes of a pair of conjugate diameters.

13. In $Ax^2 + 16y^2 + 10x + 40 = 0$ find the value of A if the slopes m_1 and m_2 of a pair of conjugate diameters are connected by the relation $m_1 m_2 = -\frac{3}{5}$.

14. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the slopes of a pair of conjugate diameters are $\frac{2}{3}$ and $-\frac{4}{5}$; find the ratio $b : a$.

147. Diameters of the hyperbola.—If in the general equation of the conic we put $A = b^2$, $B = -a^2$, $D = E = 0$, $C = -a^2b^2$ we obtain

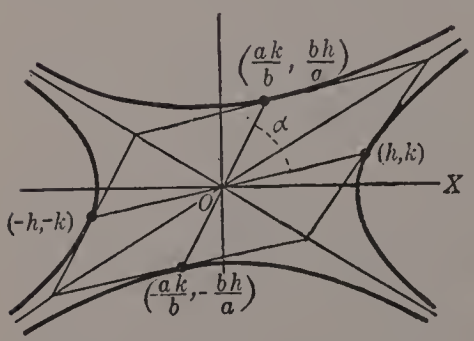
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is the equation of the hyperbola in its simplest form.

Then the equation of the diameter bisecting the chords with slope m becomes

$$y = \frac{b^2x}{a^2m}.$$

Hence every diameter of the hyperbola passes through its center. From the slopes m and $\frac{b^2}{a^2m}$ of a pair of conjugate diameters, it follows that both make acute or both make obtuse angles with the x-axis. Clearly (with the exception of those which lie on the coördinate axes) a pair of conjugate diameters cannot be at right angles to each other.



By §93 the slope of the asymptote which passes through the first quadrant is $\frac{b}{a}$. It is easily seen that if $m < \frac{b}{a}$ then $\frac{b^2}{a^2m} > \frac{b}{a}$.

Hence of a pair of conjugate diameters one makes a smaller angle with the axis than the asymptote, and the other makes a larger angle. A similar statement holds with respect to a pair of diameters lying in the second and fourth quadrants. It follows at once that if one of a pair of conjugate diameters meets the hyperbola, the other does not.

148. Conjugate diameters and conjugate hyperbolas.—If one of a pair of conjugate diameters meets the hyperbola, then the other meets the conjugate hyperbola. The length of a diameter of a hyperbola is the distance between the points where it meets the hyperbola or the distance between the points in which it meets the conjugate hyperbola.

149. Properties of diameters of the hyperbola.—Below are stated the more important properties of the diameters of the hyperbola, but the proofs are left to the student. They are very similar to the proofs of the corresponding propositions for the ellipse.

(1) If (h, k) and $(-h, -k)$ are the extremities of a diameter, then $\left(\frac{ak}{b}, \frac{bh}{a}\right), \left(-\frac{ak}{b}, -\frac{bh}{a}\right)$ are the extremities of its conjugate diameter.

(2) If a_1 and b_1 are the lengths of semiconjugate diameters of a hyperbola, then $a_1^2 - b_1^2 = a^2 - b^2$, and hence $a_1^2 - b_1^2$ is a constant.

(3) If α is the angle between two conjugate diameters, then $\sin \alpha = \frac{ab}{a_1 b_1}$.

(4) If Q_1Q_2 and R_1R_2 are a pair of conjugate diameters, then the tangents at the points Q_1, Q_2, R_1, R_2 form a parallelogram whose area is $4ab$.

EXERCISES

1. In the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ the slope of a diameter is $\frac{1}{3}$; find the extremities of the conjugate diameter.

2. Find the sine of the angle made by the pair of conjugate diameters in Ex. 1.

3. In the hyperbola $\frac{x^2}{36} - \frac{y^2}{16} = 1$ the length of a diameter is 15; find the length of the conjugate diameter.

MISCELLANEOUS PROBLEMS ON CHAPTER XII

1. Find the relation between the slopes of a pair of conjugate diameters in the equilateral hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$.

2. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ find the equations of two conjugate diameters, one of which bisects the chord joining the points $(a, 0)$ and $(0, b)$.

3. Prove that in an equilateral hyperbola two conjugate diameters make equal angles with an asymptote.

4. Prove that if on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, Q and R are extremities of a pair of conjugate diameters, then the sum of the squares of the perpendiculars from Q and R to the major axis is equal to b^2.

5. Prove that a line joining the ends of a pair of conjugate diameters of a hyperbola is parallel to one asymptote and is bisected by the other.

6. Two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are equal in length; find their equations.

7. Prove that on the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

8. Prove that the perpendicular from the focus of a parabola to a tangent meets the diameter through the point of contact on the directrix.

9. Show that in an ellipse the diameters parallel to the lines joining the extremities of the axes are conjugate.

10. Prove that chords drawn from a point on an ellipse to the extremities of a diameter are parallel to a pair of conjugate diameters.

11. Prove that the proposition of Ex. 10 holds also for a hyperbola.

12. Find the slopes of a pair of conjugate diameters of an ellipse if one of them passes through the positive end of the latus rectum.

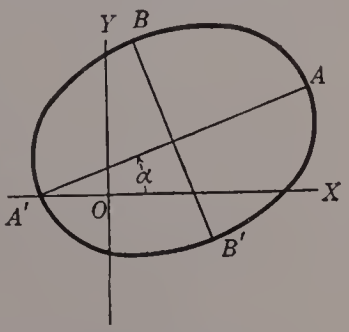
13. If a hyperbola has a pair of equal conjugate diameters, prove that the hyperbola is equilateral.

14. Prove that if a perpendicular is drawn from the focus of an ellipse to a diameter, then it will meet the conjugate diameter on a directrix.

CHAPTER XIII

ROTATION OF AXES. GENERAL SECOND DEGREE EQUATION

THE conic sections considered thus far have been placed so that their axes are coincident with, or parallel to, the coördinate axes. It is evident that a conic section may be placed so that its major or transverse axis shall make any angle α with the x-axis. By turning the axes through the angle α the new x-axis will be made parallel to the axis of the conic, and we shall have the case already studied.

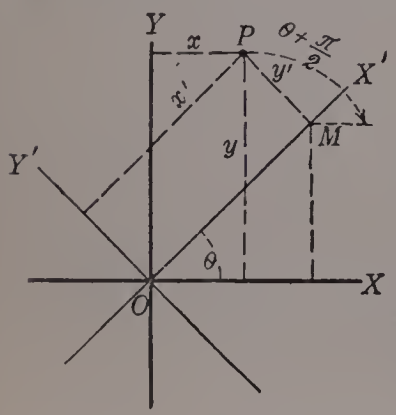


150. Turning the axes through an angle.—The rotation of axes is based on the following

THEOREM: *If the coördinates of a point are* x, y *when referred to one set of axes and* x′, y′ *are the coördinates of the same point when referred to axes through the same origin but making an angle* θ *with the first set, then*

$$x = x' \cos\theta - y' \sin.\theta$$

$$y = x' \sin\theta + y' \cos\theta \quad [47]$$



PROOF: In this proof the projection of a segment OP on the x-axis is denoted by x-proj OP and its projection on the y-axis by y-proj OP.

In the figure let x, y be the coördinates of P when referred to the old axes OX and OY, and x', y' its coördinates when referred to the new axes, OX' and OY'.

Then, using § 255.

$$x = x\text{-proj } OP = x\text{-proj } OM + x\text{-proj } MP$$

$$= OM\cos\theta + MP\cos\left(\theta + \frac{\pi}{2}\right) = x'\cos\theta - y'\sin\theta.$$

Similarly, using y-projections, prove

$$y = x'\sin\theta + y'\cos\theta.$$

151. Rotating axes to simplify the equation of a straight line.—The equation $Ax + By + C = 0$ may always be reduced to the form $y = k$ or to the form $x = h$ by turning the axes.

Example 1. Turn the axes through an angle so as to make the x-axis parallel to the line $2x - 3y + 8 = 0$.

SOLUTION: The slope of this line is $2/3$ and hence $\theta = \tan^{-1} 2/3$.

Then $$\sin\theta = \frac{2}{\sqrt{13}} \text{ and } \cos\theta = \frac{3}{\sqrt{13}}.$$

Substituting $$x = x'\cos\theta - y'\sin\theta = \frac{3}{\sqrt{13}}x' - \frac{2}{\sqrt{13}}y'$$

and $$y = x'\sin\theta + y'\cos\theta = \frac{2}{\sqrt{13}}x' + \frac{3}{\sqrt{13}}y'$$

and suppressing the primes,

$$2\left(\frac{3x}{\sqrt{13}} - \frac{2y}{\sqrt{13}}\right) - 3\left(\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}}\right) + 8 = 0,$$

or $$\frac{6x}{\sqrt{13}} - \frac{4y}{\sqrt{13}} - \frac{6x}{\sqrt{13}} - \frac{9y}{\sqrt{13}} + 8 = 0.$$

Hence $$13y = 8\sqrt{13}, \quad \text{or} \quad y = \frac{8}{13}\sqrt{13}.$$

It will be noticed that the perpendicular distance from the origin to the line remains unchanged under this transformation.

Example 2. Rotate the axes so as to make the x-axis perpendicular to the line $x + 2y - 7 = 0$.

SOLUTION: The slope of the given line is $-\frac{1}{2}$ and hence the slope of the normal from the origin upon it is 2.

Then $$\sin\theta = \frac{2}{\sqrt{5}}, \quad \cos\theta = \frac{1}{\sqrt{5}}.$$

Substituting $$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y'$$

and $$y = x'\sin\theta + y'\cos\theta = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y'$$

and suppressing the primes,

$$\frac{1}{\sqrt{5}}x - \frac{2}{\sqrt{5}}y + 2\left(\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right) = 7.$$

Hence $$\frac{5}{\sqrt{5}}x = 7, \quad \text{and} \quad x = \frac{7\sqrt{5}}{5}.$$

EXERCISES

1. Transform the equation $x^2 + y^2 = r^2$ by rotating the axes through any angle θ. Explain the result.

2. Transform the equation $xy = k^2$ by rotating the axes through the angle $\pi/4$. Compare §98.

3. Transform the equation $x^2 - y^2 = a^2$ by rotating the axes through the angle $\pi/4$ and also through the angle $-\pi/4$. Construct the figures for both problems.

4. Transform the equation $\frac{x^2}{36} + \frac{y^2}{9} = 1$ by rotating the axes through the angle $-\pi/4$. Construct the figure.

152. The general second-degree equation.—The most general equation of the second degree in two variables is

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0. \qquad (1)$$

The main problem of this chapter is to show that for any real values of A, B, H, D, E, C (A, B, and H not all zero) this equation represents a conic section.

In the excluded case, namely $A = B = H = 0$, the result is the equation $2Dx + 2Ey + C = 0$, which is not of the second degree.

When $H = 0$ the equation reduces to

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0, \qquad (2)$$

which we have already studied and found to represent a conic for all values of A, B, D, E, C.

We shall find it possible to turn the axes through an angle so as to make the coefficient of xy in the resulting equation equal to zero and thus show that (1) is reducible to (2).

This transformation will be sufficient to show that (1) represents a conic. Since the conics represented by (2) have their axes parallel to the coördinate axes, it follows that the angle through which the axes must be turned to eliminate the xy term is the angle which the transverse axis of the conic makes with one of the coördinate axes.

153. Eliminating the *xy*-term.—By §150 the axes are turned through the angle θ by the substitution

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta.$$

Substituting these values in

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0 \qquad (1)$$

and suppressing primes,

$$A(x\cos\theta - y\sin\theta)^2 + B(x\sin\theta + y\cos\theta)^2$$
$$+ 2H(x\cos\theta - y\sin\theta)\,(x\sin\theta + y\cos\theta)$$
$$+ 2D(x\cos\theta - y\sin\theta) + 2E(x\sin\theta + y\cos\theta) + C = 0,$$

or
$$x^2[A\cos^2\theta + B\sin^2\theta + 2H\sin\theta\cos\theta] +$$
$$y^2[A\sin^2\theta + B\cos^2\theta - 2H\sin\theta\cos\theta] +$$
$$2xy[-A\sin\theta\cos\theta + B\sin\theta\cos\theta + H(\cos^2\theta - \sin^2\theta)] +$$
$$x(2D\cos\theta + 2E\sin\theta) + y(2E\cos\theta - 2D\sin\theta) + C = 0.$$

The coefficient of $2xy$ is, see (5), (6), §252,

$$(B-A)\sin\theta\cos\theta + H(\cos^2\theta - \sin^2\theta) = \frac{B-A}{2}\sin 2\theta + H\cos 2\theta.$$

We now put this expression equal to zero and solve for θ.

Then, $\dfrac{B-A}{2}\sin 2\theta + H\cos 2\theta = 0$, or $\dfrac{B-A}{2}\tan 2\theta = -\,\mathrm{H}.$

Hence
$$\tan 2\theta = -\frac{2H}{B-A} = \frac{2H}{A-B},$$

or
$$\theta = \frac{1}{2}\tan^{-1}\frac{2H}{A-B} \qquad [48]$$

There is always one value of θ between 0° and 180° for which equation [48] is satisfied.

Making this substitution in (1),

$$A'x^2 + B'y^2 + 2D'x + 2E'y + C = 0.$$

Hence we have proved that by turning the axes through the angle $\dfrac{1}{2}\tan^{-1}\dfrac{2H}{A-B}$, (1) is reduced to an equation representing a conic one of whose axes is parallel to the x-axis.

154. The case when $A = B$.—The substitution indicated by [48] on the opposite page becomes meaningless when $A = B$. However, the following interpretation is consistent with mathematical principles and leads to correct results.

There is one angle between 0° and 180° which, strictly speaking, has no tangent, namely 90°. Hence $\tan^{-1}\dfrac{2H}{A-A} = \tan^{-1}\dfrac{2H}{0}$ indicates the angle with no tangent and hence indicates the angle 90°.

Therefore, in this case, $\theta = 90°/2 = 45°$.

It follows that when $A = B$ and $H \neq 0$, then one axis of the conic makes an angle of 45° with the x-axis.

155. The cases $AB - H^2 \neq 0$ and $AB - H^2 = 0$.—By using the value of $\theta = \dfrac{1}{2}\tan^{-1}\dfrac{2H}{A-B}$ in the substitution in §153 and carrying out rather obvious but somewhat lengthy operations, it may be shown that

I. If $AB - H^2 > 0$, the general second degree equation may be reduced to one of the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1), \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 \quad (2), \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0. \quad (3)$$

(1) represents an ellipse, (2) represents no real locus, and (3) represents a point.

II. If $AB - H^2 < 0$, the general equation may be reduced to one of the forms

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1), \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad (2), \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \quad (3)$$

(1) and (2) represent hyperbolas, and (3) represents two straight lines.

III. If $AB - H^2 = 0$, the general equation may be reduced to equations which represent (1) a parabola, (2) two straight lines, (3) one straight line, or (4) no locus.

Proofs are given in full on pages 176–179.

156. Reduction of numerical equations.—The general character of the locus of a second degree equation may be determined by the application of the criteria given in §155; the equation may be reduced to its simplest form by using the transformations studied in §§103, 153.

Example. What conic is represented by $2x^2 - 4xy + 6y^2 - 4x - 4y - 15 = 0$? Reduce this equation to its simplest form.

SOLUTION: (1) Since $AB - H^2 = 12 - 4 = 8$, the locus is of class I. (2) Remove first degree terms by substituting $x = x' + h$, $y = y' + k$.

Then $2(x' + h)^2 - 4(x' + h)(y' + k) + 6(y' + k)^2 - 4(x' + h) - 4(y' + k) - 15 = 0$.

Simplifying and suppressing primes, $2x^2 - 4xy + 6y^2 + x(4h - 4k - 4) + y(-4h + 12k - 4) + 2h^2 - 4hk + 6k^2 - 4h - 4k - 15 = 0$. (1)

To make the first degree terms vanish we must find values of h and k which will satisfy the equations

$$4h - 4k - 4 = 0, \quad -4h + 12k - 4 = 0.$$

Solving gives $h = 2$, $k = 1$.

Substituting in (1) we have

$$x^2 - 2xy + 3y^2 = 21/2. \tag{2}$$

Substituting

$$x = x'\cos\theta - y'\sin\theta, \; y = x'\sin\theta + y'\cos\theta$$

in (2) and suppressing primes we have

$$x^2(\cos^2\theta - 2\sin\theta\cos\theta + 3\sin^2\theta) + y^2(\sin^2\theta + 2\sin\theta\cos\theta + 3\cos^2\theta) + 2xy[2\cos\theta\sin\theta - (\cos^2\theta - \sin^2\theta)] = 21/2. \tag{3}$$

By §153 the term containing xy vanishes if the axes are turned through the angle $\theta = \frac{1}{2}\tan^{-1}\frac{-2}{1-3} = \frac{1}{2}\tan^{-1}1$. Then using (10), (11) §252,

$$\theta = \frac{45^\circ}{2}, \; \sin\theta = \sqrt{\frac{1 - \frac{1}{2}\sqrt{2}}{2}}, \; \cos\theta = \sqrt{\frac{1 + \frac{1}{2}\sqrt{2}}{2}},$$

and $\sin^2\theta = \frac{2 - \sqrt{2}}{4}$, $\cos^2\theta = \frac{2 + \sqrt{2}}{4}$, $\sin\theta\cos\theta = \frac{1}{4}\sqrt{2}$.

Substituting these values in the first line of (3),

$$(2 - \sqrt{2})x^2 + (2 + \sqrt{2})y^2 = \frac{21}{2}, \text{ or } \frac{2(2 - \sqrt{2})}{21}x^2 + \frac{2(2 + \sqrt{2})}{21}y^2 = 1.$$

The work on the opposite page shows that the given equation represents an ellipse whose semiaxes are $\sqrt{\frac{21}{2(2-\sqrt{2})}}$, $\sqrt{\frac{21}{2(2+\sqrt{2})}}$, whose center is the point (2, 1), and whose major axis makes an angle of $22\frac{1}{2}°$ with the x-axis.

It should be noticed that eliminating the first degree terms leaves the coefficients of the second degree terms unchanged, the only change being in the constant term. When the axes are turned so as to eliminate the xy term, the coefficients of both x^2 and y^2 are changed. General formulas may be found giving A' and B' (the new coefficients of x^2 and y^2) in terms of A, B, and H. These formulas are developed on pages 176 and 177.

The following exercises may be solved in the same manner as the example on page 174. In case the equation is of Class III the rotation must be performed before the translation.

EXERCISES

In each of the following determine the kind of conic represented by the equation by using the criteria given in §155. Then reduce the equation to the simplest form by means of the transformations of §§103, 153.

Group A

1. $x^2 + 4xy + 2y^2 - 6x + 4y - 6 = 0$
2. $3x^2 - 6xy + 3y^2 - 14x + 10y + 8 = 0$
3. $9x^2 - 12xy + 8y^2 + 9x + 3y - 14 = 0$
4. $8x^2 + 4xy + 2y^2 - 5x + 9y + 10 = 0$
5. $4x^2 + 2xy - 6y^2 - 6x + 10y + 4 = 0$
6. $2x^2 + 8xy + 8y^2 - 3x - 7y + 6 = 0$

Group B

1. $3x^2 - 6xy + 4y^2 - 2x + 4y - 10 = 0$
2. $6x^2 + 4xy - 2y^2 + 7x - 4y + 2 = 0$
3. $8x^2 - 4xy + y^2 - 5x + 7y - 14 = 0$
4. $3x^2 + 12xy + 12y^2 - 6x - 9y - 12 = 0$
5. $5x^2 - 8xy + 2y^2 - 3x - 5y + 16 = 0$
6. $7x^2 - 4xy + y^2 + 9x - 3y - 7 = 0$

157. Formulas for reducing second degree equations.—In the general equation

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0 \qquad (1)$$

substitute $x = x' + h$, $y = y' + k$ and suppress primes obtaining,

$$A(x+h)^2 + B(y+k)^2 + 2H(x+h)(y+k) + 2D(x+h) + 2E(y+k) + C = 0,$$

or $$Ax^2 + By^2 + 2Hxy + 2(Ah + Hk + D)x + 2(Bk + Hh + E)y + Ah^2 + 2Hhk + Bk^2 + 2Dh + 2Ek + C = 0. \qquad (2)$$

To make the x and y terms vanish, we must have

$$Ah + Hk + D = 0, \quad Bk + Hh + E = 0. \qquad (3)$$

Solving these equations simultaneously,

$$h = \frac{HE - BD}{AB - H^2}, \qquad k = \frac{HD - AE}{AB - H^2}. \qquad (4)$$

Multiplying equations (3) by h and k and adding, we have

$$Ah^2 + Bk^2 + 2Hhk + Dh + Ek = 0. \qquad (5)$$

Let $C' = Ah^2 + Bk^2 + 2Hhk + 2Dh + 2Ek + C$, and subtract from it $Ah^2 + Bk^2 + 2Hhk + Dh + Ek$ which by (5) is zero.

Then we have

$$C' = Dh + Ek + C. \qquad (6)$$

Substituting (4) in (6) we have

$$C' = \frac{ABC + 2DEH - AE^2 - BD^2 - CH^2}{AB - H^2}.$$

The quantity in the numerator is called the **discriminant** of equation (1) and is denoted by the Greek letter Δ (see §244).

Then $$C' = \frac{\Delta}{AB - H^2},$$

and equation (1) is reduced to

$$Ax^2 + 2Hxy + By^2 + C' = 0. \qquad (7)$$

By substituting $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ in (7) and suppressing primes, we have

$$(A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta)x^2 +$$
$$(A\sin^2\theta - 2H\sin\theta\cos\theta + B\cos^2\theta)y^2$$
$$+ 2[(B-A)\cos\theta\sin\theta + H(\cos^2\theta - \sin^2\theta)]xy + C' = 0. \quad (8)$$

By §153 the xy term vanishes when $\theta = \frac{1}{2}\tan^{-1}\frac{2H}{A-B}$. (9)

Equation (8) may then be written in the form

$$A'x^2 + B'y^2 + C' = 0. \quad (10)$$

The values of A' and B' may be found in terms of the original coefficients by means of (8) and (9).

That is

$$A' = A\cos^2\theta + B\sin^2\theta + 2H\sin\theta\cos\theta$$
$$= A\frac{1+\cos 2\theta}{2} + B\frac{1-\cos 2\theta}{2} + H\sin 2\theta$$
$$= \frac{1}{2}[A + B + (A-B)\cos 2\theta + 2H\sin 2\theta].$$

But by (9) $\tan 2\theta = \frac{2H}{A-B}$, and hence

$$\sin 2\theta = \pm\frac{2H}{\sqrt{(A-B)^2+4H^2}}, \quad \cos 2\theta = \pm\frac{A-B}{\sqrt{(A-B)^2+4H^2}}.$$

Therefore, $A' = \frac{1}{2}[A + B \pm \sqrt{(A-B)^2+4H^2}]$.

Similarly, $B' = \frac{1}{2}[A + B \mp \sqrt{(A-B)^2+4H^2}]$.

Hence, $A' + B' = A + B$, and $A'B' = AB - H^2$. (11)

The quantities $A + B$ and $AB - H^2$ which remain unchanged when the axes are translated or turned through an angle are called **invariants** of the quadratic function $Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C$ under transformation of the axes.

158. The case when $AB - H^2 \neq 0$.—The following results are now obvious:

1. If $AB - H^2 > 0$ and $\Delta \neq 0$, then A' and B' have the same sign and the locus is an ellipse or is imaginary; if $\Delta = 0$ the locus is a point.

2. If $AB - H^2 < 0$ and $\Delta \neq 0$, then A' and B' have different signs and the locus is a hyperbola; if $\Delta = 0$ the locus is two straight lines.

159. Central conics.—In case $AB - H^2$ is not equal to zero the general equation may be reduced to the form

$$A'x^2 + B'y^2 + C' = 0$$

whose locus, in case it is real, is symmetrical with respect to the origin. These loci are said to be **central conics,** the origin being the center. It follows that in case $AB - H^2$ is not equal to zero, the equation

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0$$

represents a central conic whose center is (h, k) where

$$h = \frac{HE - BD}{AB - H^2}, \qquad k = \frac{HD - AE}{AB - H^2}.$$

160. The case when $AB - H^2 = 0$.—Since $A'B' = AB - H^2$ it follows that if $AB - H^2 = 0$, then either $A' = 0$ or $B' = 0$. Hence, in this case, turning the axes through an angle which eliminates the xy term also eliminates one of the terms Ax^2 or By^2.

The equation is therefore reduced to the form

$$A'x^2 + 2D'x + 2E'y + C = 0.$$

If $E' \neq 0$, this equation represents a parabola. If $E' = 0$, it represents two parallel lines, one line, or no locus.

We will now show how the value of Δ may be used to distinguish between these cases.

Solving for x the equation

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0$$

$$x = \frac{-2Hy - 2D \pm \sqrt{(2Hy + 2D)^2 - 4A(By^2 + 2Ey + C)}}{2A}.$$

$$= \frac{-Hy - D \pm \sqrt{(H^2 - AB)y^2 + 2(DH - AE)y + D^2 - AC}}{A}.$$

Now note that $\Delta = ABC + 2DEH - AE^2 - BD^2 - CH^2$

$$= \frac{(H^2 - AB)(D^2 - AC) - (DH - AE)^2}{A}.$$ See §§ 244, 245.

Hence if $H^2 - AB = 0$ and $\Delta = 0$, then $DH - AE = 0$; and if $H^2 - AB = 0$ and $\Delta \neq 0$, then $DH - AE \neq 0$.

Hence if $\Delta \neq 0$ and $H^2 - AB = 0$,

$$x = \frac{-Hy - D \pm \sqrt{2(DH - AE)y + D^2 - AC}}{A}. \qquad (1)$$

Clearly (1) has a real locus, which by the above is a parabola.

If $\Delta = 0$, then, since $DH - AE$ is also 0,

$$x = \frac{-Hy - D \pm \sqrt{D^2 - AC}}{A}. \qquad (2)$$

If $D^2 - AC > 0$, (2) represents two parallel lines; if $D^2 - AC = 0$, (2) represents one straight line; and if $D^2 - AC < 0$, it has no locus.

Hence we have that if $AB - H^2 = 0$ and $\Delta \neq 0$, the equation represents a parabola; if $AB - H^2 = 0$ and $\Delta = 0$, it represents two parallel lines, one line, or no locus, according as $D^2 - AC$ is greater than, equal to, or less than zero.

161. Classification of conics.—Denoting $AB - H^2$ by K, and $ABC + 2DEH - AE^2 - BD^2 - CH^2$ by Δ (see page 176), the following table shows the classification of the loci of the equation

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0.$$

	$\Delta \neq 0$	$\Delta = 0$
$K > 0$	Ellipse or no locus	A point
$K < 0$	Hyperbola	Two straight lines
$K = 0$	Parabola	Two straight lines, one line, or no locus According as $D^2 - AC \gtreqless 0$

EXERCISES

Find the kind of curve represented by each of the following equations Find the direction of the axes and plot the curves. Reduce each equation to the simplest form.

Group A

1. $x^2 - 4xy - 2y^2 - 14x + 4y + 25 = 0$
2. $x^2 + 2xy + 2y^2 - 6x - 2y + 9 = 0$
3. $x^2 + 6xy + 9y^2 + 2x - 6y + 8 = 0$
4. $6x^2 - 5xy - 6y^2 - 46x - 9y + 60 = 0$
5. $4x^2 + 9y^2 + 4x + 1 = 0$
6. $4xy + 3y^2 - 8x + 16y + 12 = 0$
7. $9x^2 - 12xy + 4y^2 - 18x + 12y + 34 = 0$
8. $5x^2 - 13xy + 5y^2 + 10x - 13y + 42 = 0$
9. $5x^2 + 2xy + 5y^2 - 12x - 12y = 0$

Group B

1. $3x^2 - 10xy + 8y^2 - 15x + 7y - 16 = 0$
2. $11x^2 - 4xy + 14y^2 - 13x + 12y + 15 = 0$
3. $3x^2 + 10xy + 3y^2 - 2 = 0$
4. $x^2 - 2xy + y^2 + 8x - 10y + 2 = 0$
5. $x^2 + 4y^2 - 2x - 16y - 14 = 0$
6. $17x^2 - 12xy + 22y^2 - 26 = 0$
7. $41x^2 - 24xy + 34y^2 - 188x + 116y + 196 = 0$
8. $9x^2 + 24xy + 16y^2 - 80x + 60y + 16 = 0$
9. $4x^2 - 8xy + 4y^2 + 6x - 8y + 4 = 0.$

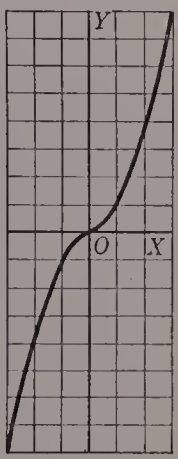
Group C

1. $x^2 - y^2 + \sqrt{3}xy + \sqrt{3}x + \sqrt{3}y + 2 + \sqrt{3} = 0$
2. $3x^2 + y^2 + 2xy - 2x + 2y + 11 = 0$
3. $x^2 + 4xy - 3y^2 - 6x - 12y + 9 = 0$
4. $x^2 - xy - y^2 - 3x - 6y - 9 = 0$
5. $x^2 + y^2 - 2xy + 2x - 10y + 16 = 0$
6. $2x^2 + xy - y^2 + 4x + 11y - 6 = 0$
7. $2x^2 + 4xy - 3y^2 + 10y - 5 = 0$
8. $3x^2 + 3y^2 - 7xy - 38x + 40y + 116 = 0$
9. $3x^2 + 3y^2 + 3xy - 8x + 16y + 20 = 0$

CHAPTER XIV

HIGHER PLANE CURVES

With the exception of a few examples studied in Chapter IX, the curves considered thus far are loci of equations of the first and second degree. We will now study some other plane curves that are of particular interest.



162. Curves of the type $y = x^n$.—We have already studied the parabola $y = x^2$. The following examples indicate the character of the loci of $y = x^n$ for higher values of n.

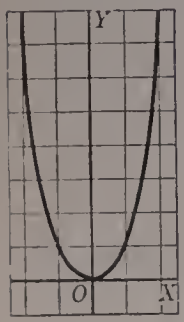
Example 1. Construct the locus of $y = x^3$.

The following statements are obvious:

(a) y is positive for all positive values of x, negative for all negative values of x, and zero for $x = 0$.

(b) The curve lies below the parabola $y = x^2$ for values of x between $x = 0$ and $x = 1$, cuts this parabola for $x = 1$, and lies above it for values of x greater than 1.

(c) The part of the curve which lies in the third quadrant is of the same shape as that which lies in the first quadrant.



(d) The curve is symmetrical with respect to the origin.

Example 2. Construct the locus of $y = x^4$.

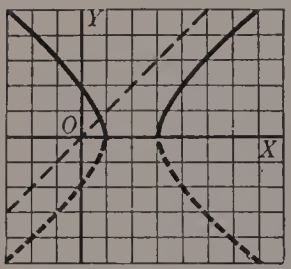
Discuss this curve as to

(a) symmetry; (b) comparison with $y = x^3$ for x between $x = 0$ and $x = 1$, and for x greater than 1.

Example 3. Discuss the locus of $y = x^n$, (a) as to relative location in the first quadrant for different values of n, (b) as to location in the second or third quadrants, (c) as to symmetry. Distinguish the cases when n is even and when n is odd.

163. Loci representing the square roots of polynomials.—Consider the following examples:

Example 1. Describe the locus of the equation $y = \sqrt{(x-1)\ (x-3)}$ Note that the radical sign indicates positive values only.



(a) For what values of x is y real and for what values of x is it imaginary? For what values of x is it zero?

(b) For $x>3$ is $y<x$ or $y>x$?

(c) For $x<\frac{3}{4}$ is $y>|x|$ or $y<|x|$?

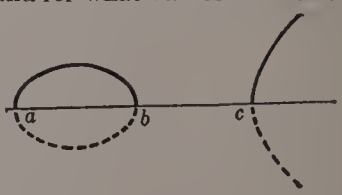
(d) Find pairs of values of x and y which satisfy the equation and sketch the curve.

(e) How is the curve changed if the equation is made to read $y^2 = (x-1)\ (x-3)$? Sketch this curve.

Example 2. Describe the locus of $y = \sqrt{(x-a)\ (x-b)\ (x-c)}$ if $a<b<c$.

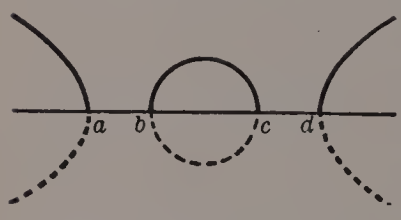
(a) For what values of x is y equal to zero and for what values is it real?

(b) As x takes large positive values is the value of y greater or less than x? Is it greater or less than x^2?



(c) How is the curve changed if the equation is made to read $y^2 = (x-a)\ (x-b)\ (x-c)$?

Example 3. Describe the locus of $y = \sqrt{(x-a)\ (x-b)\ (x-c)\ (x-a)}$ if $a<b<c<d$.



(a) For what values of x is y real, zero, imaginary?

(b) As x takes large positive values is y greater or less than x? greater or less than x^2? greater or less than $x^{2/3}$?

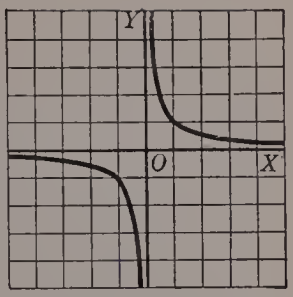
(c) Answer similar questions for x taking large negative values.

(d) How is the curve changed if the equation is made to read $y^2 = (x-a)\ (x-b)\ (x-c)\ (x-d)$?

Example 4. Describe the locus of $y^2 = (x-a)\ (x-b)\ (x-c)\ (x-d)\ (x-e)$ if $a<b<c<d<e$.

164. Loci representing fractional functions.—The equation $y = \frac{k}{x}$ (or $xy = k$) has already been studied (§§92, 98). The

locus of this equation possesses one of the chief characteristics of all such loci.



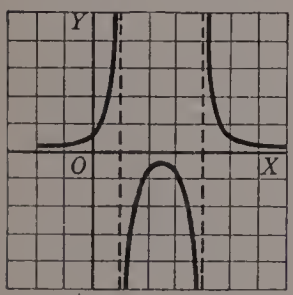
Example 1. In $y = \frac{1}{x}$ describe the change in y as

(a) x takes very large positive values.
(b) x takes very large negative values.
(c) x approaches zero from the positive side.
(d) x approaches zero from the negative side.

Note that for $x = 0$ the function $\frac{1}{x}$ has no value since division by zero does not give a number. This function is said to have an **infinite discontinuity** at $x = 0$.

Example 2. Describe the locus of $y = \frac{1}{(x-1)(x-4)}$.

Is y positive or negative for $x > 4$? for $1 < x < 4$? for $x < 1$?



How does y change

(a) As x takes large positive values?

(b) As x approaches 4 from the positive direction?

(c) As x approaches 4 from the negative direction?

(d) As x approaches 1 from the positive direction?

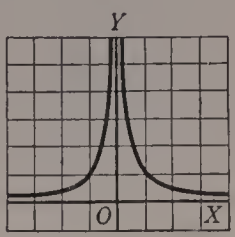
(e) As x approaches 1 from the negative direction?

(f) As x grows large in the negative direction?

Example 3. Describe the locus of $y = \frac{1}{x^2}$.

(a) How does y vary as x takes very large values either positive or negative?

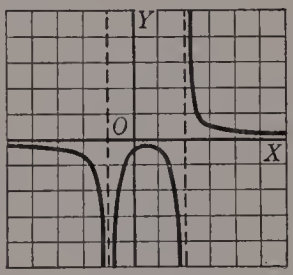
(b) How does y vary as x approaches zero from the positive side? as it approaches zero from the negative side?

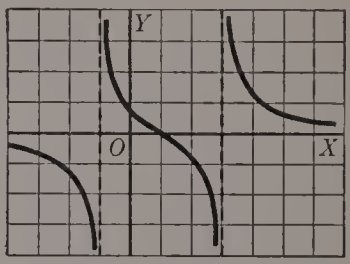


Clearly this function is not defined for $x = 0$. It has an infinite discontinuity at $x = 0$ although y grows infinitely large in the same direction (positive) when x approaches zero from either side.

Example 4. Describe the locus of the equation $y = \dfrac{1}{(x+1)^2\,(x-2)}$.

State and answer questions similar to those given in the preceding examples.





Example 5. Describe the locus of the equation $y = \dfrac{1}{x+1} + \dfrac{1}{x-3}$.

(a) For what values of x is y equal to zero?

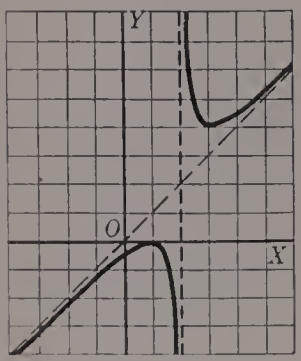
(b) For what values of x does this function have infinite discontinuities?

(c) How does y vary as x takes very large values either positive or negative?

(d) How does y vary as x approaches 3 from the positive direction? from the negative direction?

Example 6. Describe the function $y = \dfrac{x^2 - 2x + 1}{x - 2}$.

SUGGESTIONS: Reduce to $y = x + \dfrac{1}{x-2}$.



Construct the line $y = x$. Then add to y the ordinates in $y = \dfrac{1}{x-2}$.

(a) For what values of x does this function have an infinite discontinuity?

(b) How does y vary as x takes large values either positive or negative?

(c) For what values of x is this function equal to zero?

165. Loci of irrational algebraic functions. — The loci studied in §163 are irrational algebraic curves. The following are additional examples of this class.

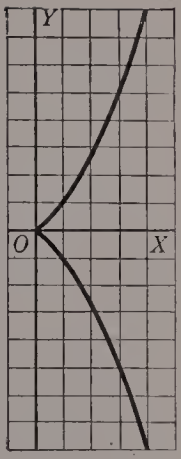
Example 1. Construct the locus of $y = x^{3/2}$.

Assume that $x^{3/2}$ represents positive values only.

(a) For what values of x is y real?

(b) Does this curve lie above or below the line $y = x$? Answer this question for $0 < x < 1$, and for $x > 1$.

(c) How is the curve changed if the equation is made to read $y^2 = x^3$?



$y^2 = x^3$

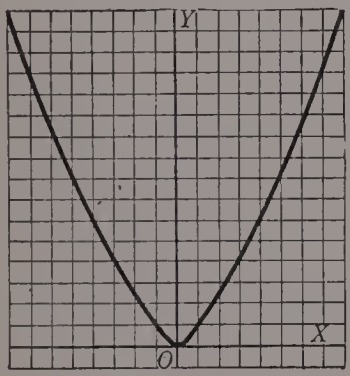
Example 2. Construct the locus of $y^3 = x^4$. Compare this curve with $y = x^2$.

Example 3. Construct the locus of $x^{1/2} + y^{1/2} = a^{1/2}$.

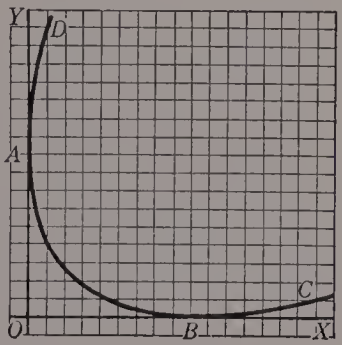
(a) Show that neither x nor y can be negative. In what quadrant does the curve lie? (In the figure $a = 9$.)

(b) If $x^{1/2}$ and $y^{1/2}$ denote positive square roots what is the greatest possible value of x? of y?

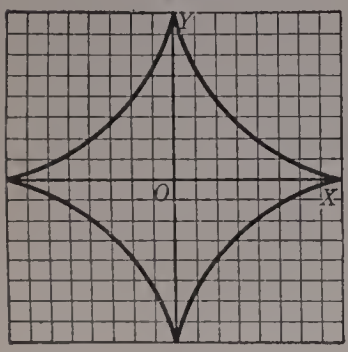
(c) Show that in the figure the arc AB is the locus of $x^{1/2} + y^{1/2} = a^{1/2}$, the branch $BC\ldots$ is the locus of $x^{1/2} - y^{1/2} = a^{1/2}$, and the branch $AD\ldots$ is the locus of $y^{1/2} - x^{1/2} = a^{1/2}$.



$y^3 = x^4$



$\pm x^{1/2} \pm y^{1/2} = a^{1/2}$



Example 4. Consider the locus of $x^{2/3} + y^{2/3} = a^{2/3}$.

(a) Is the locus symmetrical with respect to the x-axis? with respect to the y-axis?

(b) Can x have a value greater than a? less than $-a$? Answer similar questions for y.

(c) Construct the curve for $a = 8$.

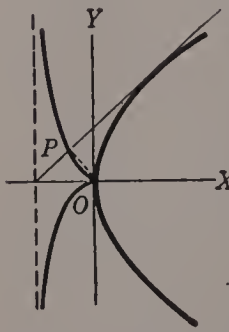
166. Intersection with tangent of normal from origin upon tangent to conic.—In these examples the equations of the parabola, ellipse, and hyperbola are in the forms

$$y^2 = 4px, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Example 1. Find the locus of the intersection of a tangent to the parabola $y^2 = 4px$ and the normal from the origin upon this tangent.

SOLUTION: Let the slope of the tangent be m. Then its equation is (§60)

$$y = mx + p/m, \tag{1}$$



and the equation of the normal upon it through the origin is

$$y = -\frac{1}{m}x, \quad \text{whence } m = -\frac{x}{y}.$$

Substitute this value of m in (1) and reduce to

$$y^2 = -\frac{x^3}{x+p}, \tag{2}$$

which is the equation of the required locus.

Then, (a) for what values of x is y real?

(b) How does y vary as x approaches $-p$ from the positive direction? What line is an asymptote to the curve?

(c) Discuss this curve as to symmetry.

(For another construction of this curve see page 188.)

Example 2. Find the locus of the intersection of a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the normal through the origin upon this tangent.

SOLUTION: If m is the slope of the tangent then its equation (§75) is

$$y = mx + \sqrt{a^2m^2 + b^2} \tag{1}$$

and the equation of the normal upon it through the origin is $y = -x/m$. Substitute $m = -x/y$, in (1) and reduce to

$$(x^2 + y^2)^2 = a^2x^2 + b^2y^2.$$

Construct the locus of this last equation for $a = 2$ and $b = 1$.

SUGGESTION: Find the values of y for $x = \frac{1}{2}$, $x = 1$, $x = 1\frac{1}{2}$.

Note that (0, 0) is a point on this locus. Does this point satisfy the conditions of the problem?

Example 3. Find the locus of the intersection of a tangent to the rectangular hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

and the normal through the origin upon this tangent.

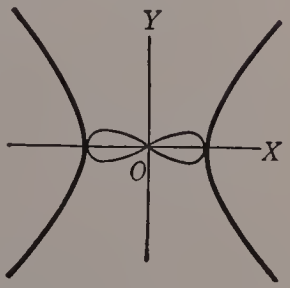
SOLUTION: If m is the slope of the tangent then its equation is (§89),

$$y = mx + \sqrt{a^2m^2 - a^2} \qquad (1)$$

and the normal upon it through the origin is $y = -\frac{1}{m}x$, whence $m = -\frac{x}{y}$

Remove the radical from (1) by transposing and squaring, and substitute $m = -\frac{x}{y}$ obtaining,

$$(x^2 + y^2)^2 = a^2(x^2 - y^2). \qquad (2)$$



(a) For what values of x is $y = 0$ in this equation?

(b) For what values of x is y imaginary?

(c) Consider the locus of this equation as to symmetry.

(d) Construct the locus for $a = 2$.

SUGGESTION: Find values of y for $x = \frac{1}{2}$, $x = 1$, $x = \frac{3}{2}$.

This curve is called the **lemniscate** of **Bernoulli.** Its general form is shown in the figure by the two ovals between the two branches of the hyperbola.

167. The Ovals of Cassine.—Find the locus of a point which moves so that the product of its distances from two fixed points is a constant. Study the locus for different values of the given constant.

SUGGESTIONS: Let $(-a, 0)$, $(a, 0)$ be the fixed points, and c^2 the given product. Show that the equation of the locus is

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2) + c^4 - a^4.$$

Solve this equation for y^2.

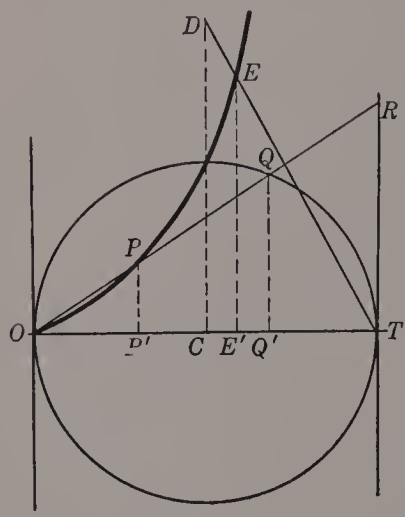
Study the curve (a) for $c^2 = a^2$, (b) for $c^2 > a^2$, (c) for $c^2 < a^2$.

Note that for $c^2 = a^2$ the equation reduces to

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2).$$

Compare with the equation obtained in Example 3, §166.

168. The Cissoid of Diocles.—On the segment OT (on the x-axis) as a diameter construct a circle, and draw a tangent to it at T. OR is any line through O meeting the circle in Q and the tangent in R. On OR lay off OP equal in length to QR. Find the locus of P.



SOLUTION: The coördinates of P are $x = OP'$, $y = P'P$. Let r = radius of circle.

Since $OP = QR$ it follows that $OP' = Q'T$.

Then $\dfrac{y}{x} = \dfrac{Q'Q}{OQ'} = \dfrac{Q'Q}{2r - x}$.

But $\overline{Q'Q}^2 = OQ' \cdot Q'T = (2r - x)x$.

Hence $\dfrac{y}{x} = \pm \dfrac{\sqrt{(2r - x)x}}{2r - x}$,

or $y^2 = \dfrac{x^3}{2r - x}$. See §239.

(Note that the locus of this equation is symmetrical with respect to the x-axis, and that the figure shows only the upper half of it.)

This equation is identical to that obtained in Example 1 page 186, when p is replaced by $-2r$. Hence if the parabola used in that example were $y^2 = -8rx$, we should get exactly the equation just found.

The locus of this equation is called the **Cissoid of Diocles.**

The cissoid may be used to "duplicate the cube" as follows:

Through C, the center of the circle, draw a line parallel to the y-axis and on it lay off $CD = 2r$. Draw DT meeting the cissoid in E, and draw $E'E$ parallel to CD.

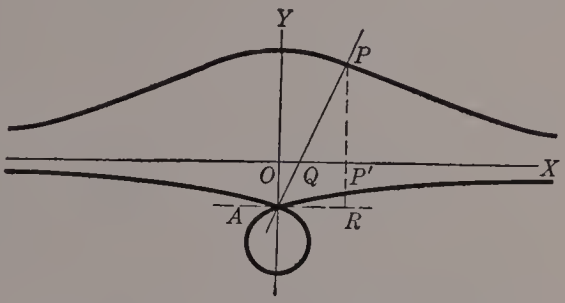
Then $\overline{E'E}^3 = 2\overline{OE'}^3$ for by the equation of the cissoid

$$\overline{E'E}^2 = \frac{\overline{OE'}^3}{\overline{E'T}} = \frac{\overline{OE'}^3}{\frac{1}{2}\,\overline{E'E}} \quad \text{or} \quad \overline{E'E}^3 = 2\overline{OE'}^3.$$

Note that if $CD = mr$, then $E'T = \dfrac{1}{m}E'E$ and hence $\overline{E'E}^3 = m \cdot \overline{OE'}^3$.

The cissoid may, therefore, be used to multiply the volume of the cube by any number.

169. The Conchoid of Nicomedes.—Let A be a point on the y-axis (placed below the origin). Through A a line is drawn meeting the x-axis in Q. On AQ lay off $QP = b$, where b is a constant for all lines AQ. QP is laid off in either direction from Q. Find the locus of P.



SOLUTION: Let $AO = a$. If x, y are the coördinates of P, then $OP' = x$ and $P'P = y$.

Now, $AR : RP = QP' : P'P.$

Also, $QP' = \sqrt{b^2 - y^2}.$

Hence, $x : y + a = \sqrt{b^2 - y^2} : y,$

from which we have $x^2y^2 = (y + a)^2(b^2 - y^2).$

Since QP may be taken in the direction QA, y may be negative. Both cases are included in the equation. The figure shows the curve for $a < b$.

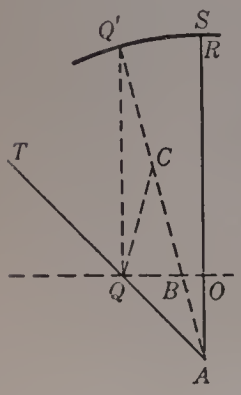
This curve is called the **Conchoid of Nicomedes.** The point A is the **pole** of the conchoid and the line OP' is its **directrix.**

(a) Consider the locus as to symmetry.

(b) What asymptotes, if any, does this curve have?

(c) Construct the branch of the curve below the x-axis, (1) when $b < a$, (2) when $b = a$, (3) when $b > a$.

This curve may be used to "duplicate the cube," and also to trisect an angle.

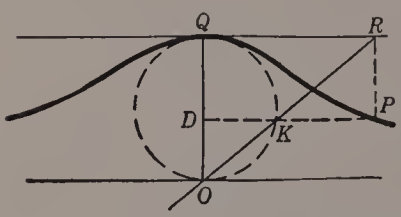


To trisect an angle SAT, let Q be any point on AT. Through it draw a line perpendicular to AS, meeting it in O. Take $OR = 2AQ$. Use OQ as the directrix and A as the pole and construct a conchoid passing through the point R. QQ' is perpendicular to OQ, and C is the middle point of BQ' in the figure. Then $AQ = BC = QC = CQ'$.

Hence $\angle CQ'Q = \frac{1}{2}\angle BCQ = \frac{1}{2}\angle BAQ$. But $\angle CQ'Q = \angle OAB$.

Hence $\angle OAB = \frac{1}{2}\angle BAQ$, and the line AB trisects the given angle SAT.

170. The Witch of Agnesi.—On $OQ = 2r$ (lying on the y-axis) as a diameter, construct a circle and at Q draw a tangent QR to the circle. From O draw a line meeting the circle in K and the tangent in R. Through K and R draw lines parallel respectively to the x-axis and y-axis, and meeting in P.



The locus of P is the **Witch of Agnesi.**

Let KP meet the diameter OQ in D. If x, y, are coördinates of P, then $x = DP$, $y = OD$.

But $OD : OQ = DK : QR$, or $y : 2r = \sqrt{y(2r - y)} : x$. (See § 239).

Hence, $x^2y = 4r^2(2r - y)$,

which is the equation of the Witch.

(a) Consider this locus as to symmetry.

(b) For what values of x is y real?

(c) For what values of y is x real?

(d) What asymptotes, if any, does this curve have?

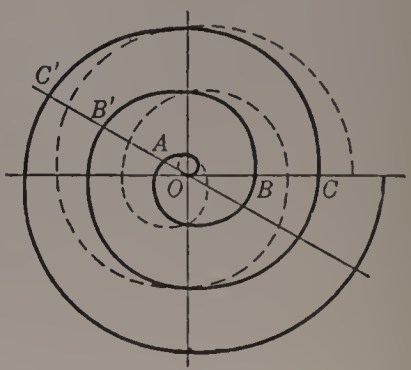
171. Spirals.—The examples given below are the more important of a class of curves called spirals. Polar coördinates are particularly adapted for the equations of these curves.

Example 1. The spiral of Archimedes. If ρ, θ are the polar coördinates of P and if

$$\rho = c\theta$$

where c is a constant, then P describes the spiral of Archimedes.

Let OC be the polar axis and O the origin. If the axis cuts the curve at B, C, . . . then $OB = 2\pi c$, $BC = 2\pi c$, . . . Similarly, on the line OC', $AB' = B'C' = 2\pi c$.



To construct the curve draw lines through the origin making angles $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{5\pi}{6}$, . . . with the initial line,

and on each line mark the points in which the curve will cut it. Then sketch the curve. Note how points of the type A must be constructed.

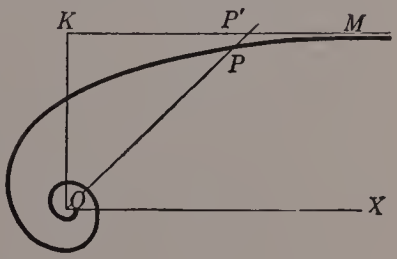
Discuss the curve for negative values of θ.

Example 2. The so-called hyperbolic spiral is defined by the equation

$$\rho = \frac{c}{\theta}, \quad \text{or } \rho\theta = c.$$

It is clear that ρ increases indefinitely as θ approaches zero.

Let OX be the polar axis. Construct OK perpendicular to OX and equal to c; construct KM perpendicular to OK. Let any radius vector meet the curve in P and the line KM in P'.



Then $OP' = \dfrac{c}{\sin\theta}$. Since $\sin\theta < \theta$, $OP < OP'$ and the curve lies below the line KM. Denote OP' by ρ'.

Then $\rho : \rho' = \dfrac{c}{\theta} : \dfrac{c}{\sin\theta} = \dfrac{\sin\theta}{\theta}$.

Since $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$, $\lim\limits_{\theta \to 0} \dfrac{\rho}{\rho'} = 1$

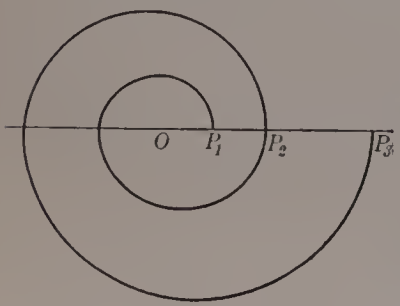
and the curve approaches KM as an asymptote as θ approaches zero.

As θ increases indefinitely ρ approaches zero and the curve makes an infinite number of turns around the origin without ever reaching it.

Discuss this curve for negative values of θ.

Example 3. The logarithmic spiral is defined by the equation

$$\rho = c^{c\theta}.$$



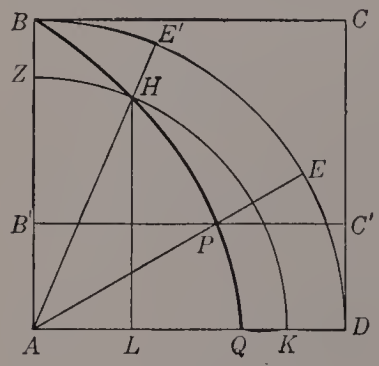
When $\theta = 0$, $\rho = 1$. As θ increases the curve winds around the polar origin as shown in the figure. When θ is negative and increases numerically without limit, ρ approaches zero.

If $P_1, P_2, P_3, \ldots$ are points in which this curve cuts the polar axis successively as θ increases from zero, what is the relation between the segments $OP_1, P_1P_2, P_2P_3, \ldots$?

In the figure shown here c is taken so that

$$c^{2c\pi} = \frac{9}{4}.$$

172. The quadratrix.—The (so-called) **quadratrix** which was invented by the Greeks for the purpose of constructing the area of the circle, depends on two uniform motions, one angular and the other linear.



In a square $ABCD$ the side AB retaining a constant length is made to swing at a uniform rate about A as a center, the point B describing the arc BED. The side BC moves down at a uniform rate, remaining parallel to AD. The motions begin and end at the same time, AE and BC coinciding with AD at the same moment. If AE and $B'C'$ are simultaneous positions of these lines, then their intersection point P describes the quadratrix.

If Q is the point of intersection of the quadratrix with AD, then it may be proved that arc $BED : AD = AD : AQ$.

The proof is in part as follows: Suppose this proposition is not true and that there is some point K between Q and D such that

$$\text{arc } BED : AD = AD : AK. \tag{1}$$

Construct arc $KIIZ$ with A as center, II being on the quadratrix.

Since $$\text{arc } BED : \text{arc } ZIIK = AD : AK,$$

then from (1) $$\text{arc } ZIIK = AD = BA. \tag{2}$$

Also, $$\text{arc } BED : \text{arc } E'D = BA : IIL \tag{3}$$

Since arc BED and arc $ZIIK$ are concentric,

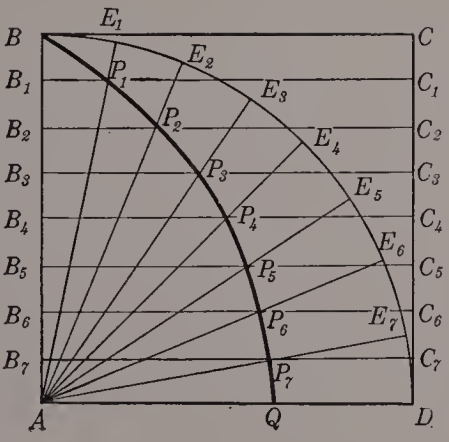
$$\text{arc } BED : \text{arc } E'D = \text{arc } ZIIK : \text{arc } IIK = BA : \text{arc } IIK. \tag{4}$$

From (3) and (4) $IIL = \text{arc } IIK$ which is impossible.

The proportion, arc $BED : AD = AD : AQ$, enables us to construct a straight line equal in length to the arc BED. Since the area of a circle is equal to a rectangle whose base is the length of the circle and the altitude equal to half the radius, a square whose area is equal to that of the circle may be constructed.

The quadratrix may be constructed as follows:

Suppose the arc BD is divided into eight equal parts by the points $E_1, E_2, E_3, \ldots$ and the line BA is divided into eight equal parts by the points $B_1, B_2, B_3, \ldots$ Draw the radii $AE_1, AE_2, AE_3, \ldots$ and the lines $B_1C_1, B_2C_2, B_3C_3, \ldots$ parallel to BC. Let AE_1 and B_1C_1 meet in P_1, AE_2 and B_2C_2 in P_2, and so on. Then the points $B, P_1, P_2, \ldots$ lie on the required curve and a smooth curve drawn through these points is the quadratrix. Since the purpose of the construction is to find the point Q where the curve meets AD, only the lower part of the curve needs to be constructed.



While all the points $P_1, P_2, \ldots$ may be located by using ruler and compass, the curve through them cannot be thus "constructed." No matter how close to the line AD these points may be, the curve through them must be "sketched." Hence this construction does not satisfy the requirement that the area shall be constructed by means of these instruments. It is of course well known that this cannot be done. However, this construction enables us to construct a square whose area shall be as near to that of a given circle as we may please.

The equation of the quadratrix is obtained as follows: Let (ρ, θ) be any point on it and let $AB = r$, A being the pole and AD the polar axis.

Then $$\rho \sin\theta : r = \theta : \frac{\pi}{2}$$

and hence $$\rho = \frac{2r\theta}{\pi \sin\theta}.$$

From this equation it is easily verified that arc $BD : AD = AD : AQ$.

Using this curve, any angle may obviously be divided into any number of equal parts. Let P be any point on the curve. Draw PP' parallel to AD meeting AB in P'. Divide AP' into the desired number of equal parts and through the points of division draw lines parallel to AD. Join to A the points in which these lines meet the curve.

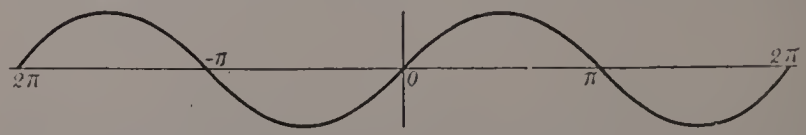
173. Loci of trigonometric functions.—In this section we will consider the loci of the sine function and the cosine function. Graphic representation of the other trigonometric functions will be found in texts on trigonometry.

Example 1. Construct the graphs of $y = \sin x$ and $y = \cos x$.

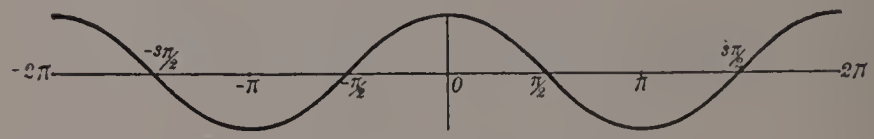
Reducing angles to radian measure and using a table of natural functions we find for the following values of the sines of the angles 0°, 15°, 30°, . . .

x	0.26	0.52	0.79	1.05	1.31	1.57	1.83	2.09
y	0.26	0.50	0.71	0.87	0.97	1.00	0.97	0.87

Constructing such a table from $x = -2\pi$ to $x = 2\pi$, plotting the corresponding points, and drawing a smooth curve through them, we have a curve of the general form shown here.

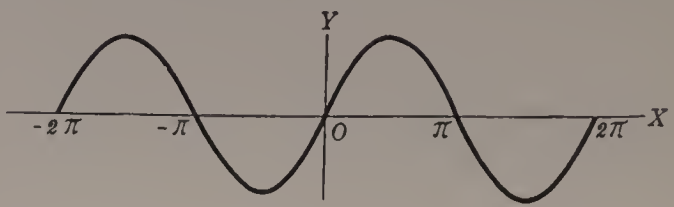


Constructing the locus of $y = \cos x$ in a similar manner we have a curve like the following



Example 2. Construct the locus of the equation $y = 2 \sin x$.

It is obvious that the locus is exactly like the sine curve shown above with the exception that every ordinate has been multiplied by 2. The locus is shown in the figure below.



Example 3. Construct the locus of $y = \sin 2x$.

This locus is easily found from the sine curve on the page opposite. Since the angle in this case is $2x$ while in $\sin x$ it is x, it follows that for a given value x_1 of x, $\frac{x_1}{2}$ gives the same value of $\sin 2x$ as x_1 gives of $\sin x$. Hence the arches of $\sin 2x$ are only half as wide as those of $\sin x$.

Example 4. Discuss the locus of $y = a \sin bx$ for different values of a and b.

(a) Suppose $b = 1$. How are the arches of the curve affected by making a very small? by making a very large?

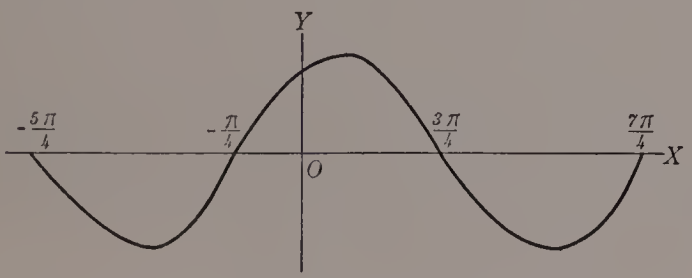
(b) Suppose $a = 1$. How are the arches of the curve affected by making b very large? by making b very small?

(c) In general, how is the locus of $y = a \sin bx$ affected by changing the value of a? by changing the value of b?

Example 5. Construct the locus of $y = \sin x + \cos x$. Finding values of $\sin x$ and $\cos x$ as on the opposite page and adding them we have

x	0	0.26	0.52	0.79	1.05	1.31	1.57	1.83	2.09	2.36
y	1	1.23	1.37	1.42	1.37	1.23	1.00	0.71	0.37	0

The general form of the curve is shown below.

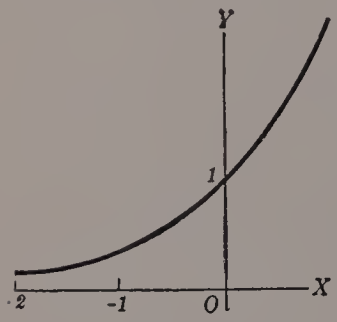


It is clear that the locus of $y = a_1 \sin b_1 x + a_2 \sin b_2(x - c)$ may be constructed for any given values of a_1, b_1, a_2, b_2, c. Note that substituting $x - c$ for x translates $a_2 \sin b_2 x$ a distance c along the x-axis (see page 121). Curves of the type $y = a \sin bx$ are used to represent simple wave motions while curves of the type $y = a_1 \sin b_1 x + a_2 \sin b_2(x - c)$ are used to represent the resultant when two wave motions of different wave lengths and different intensities are superimposed.

174. Exponential and logarithmic functions.—A function in which the independent variable occurs in an exponent is called an exponential function.

Thus, 2^x and a^{kx} (a and k being constants), are exponential functions.

The number $e = 2.718281829\ldots$ used as a base gives an exponential function of special importance.



Example 1. Construct the locus of $y = e^x$.

SOLUTION: Using a table of logarithms the following pairs of values are found:

x	0	.2	.4	.6	.8	1	$-.5$	-1	-2
y	1	1.22	1.49	1.82	2.23	2.72	.61	.37	.14

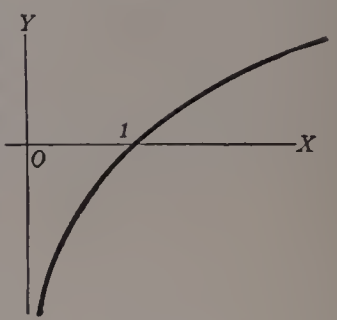
The general shape of the curve is shown in the figure.

(a) Describe the curve as x becomes numerically large in the negative direction. Is y negative for any value of x?

(b) Show that for any fixed positive value n_1 of n there is a value x_1 of x such that $e^{x_1} < x_1^{n_1}$.

(c) Substantiate the following statement: As x increases indefinitely e^x becomes greater than x^n for every value of n.

When logarithms are used for theoretical purposes e is used as the base. The expression $\log x$ is then often used to indicate $\log_e x$.



Example 2. Construct the locus of $y = \log_e x$.

SOLUTION: If $y = \log_e x$, then $x = e^y$. Hence the graph may be constructed by interchanging the values of x and y in the table given above. The general form of the locus is shown in the figure.

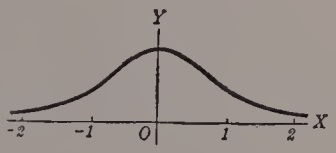
Find a line such that with respect to it the figure in Example (1) is symmetrical to (2).

Each of the functions e^x and $\log_e x$ is the inverse of the other. Note how their loci are related to the coördinate axes.

Example 3. Construct the locus of $y = e^{-x^2}$.

SOLUTION: First construct the part of the curve which lies in the first quadrant. Using a table of natural logarithms the following set of values may be found.

x	0	.2	.4	.6	.8	1	2
y	1	.96	.85	.70	.53	.37	.02



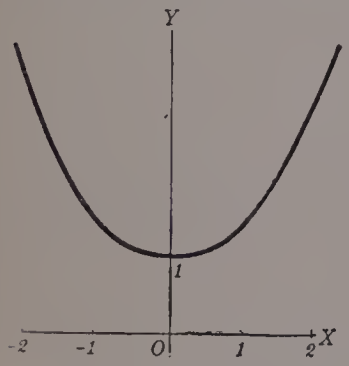
(a) Discuss this curve as to symmetry.

(b) What asymptotes, if any, does this curve have?

Example 4. Compare the locus of $y = ax^{-kx^2}$ with the locus of $y = e^{-x^2}$, (a) when $k = 1$ and a takes different positive values; (b) when a is a fixed number and k takes different positive values.

The curve $y = ae^{-kx^2}$ is called the **probability curve**; it is important in the theory of probability and in the mathematical theory of statistics.

Example 5. Construct the locus of $y = \frac{1}{2}(e^x + e^{-x})$.



First construct the curve for positive values of x.

Using a table of logarithms the following values may be found.

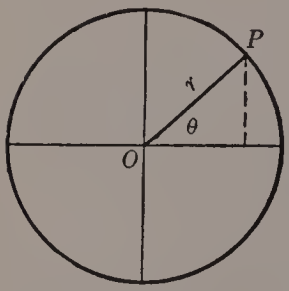
x	0	.2	.4	.6	.8	1	2
y	1	1.02	1.08	1.18	1.34	1.55	3.77

Discuss this curve as to symmetry.

Example 6. Compare the locus of $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ with the locus of $y = \frac{1}{2}(e^x + e^{-x})$ for various positive values of a.

The locus of $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ is called the **catenary** and is the curve in which a chain of uniform weight will hang when supported at both ends.

175. Parametric representation.—Thus far we have used a single equation to express the relation between the coördinates of the points on a curve. Thus $x^2 + y^2 = r^2$ is the relation between the coördinates on the circle with radius r and center at the origin.



We will now study a different way of representing such relations.

If $P(x, y)$ is any point on the circle and θ the angle which the radius OP makes with the x-axis, then

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In these equations any value of θ determines a value of x and of y. That is, for every value of θ a definite point P is fixed by means of these equations.

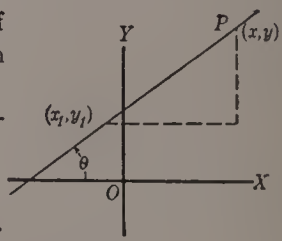
More generally, if it is given that

$$x = f_1(\theta), \quad y = f_2(\theta),$$

where $f_1(\theta)$ and $f_2(\theta)$ are definite functions of θ, then every value of θ will determine a value for each of the coördinates, and hence a fixed point. In general all such points will lie on a curve.

In these equations θ is called a **parameter,** and the two equations constitute a **parametric representation** of the curve.

Example 1. Derive parametric equations of the straight line through (x_1, y_1) which makes an angle θ with the x-axis.

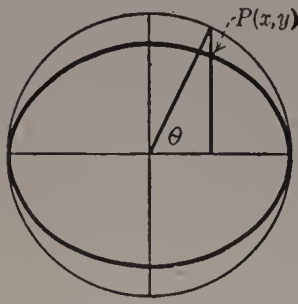


It is left for the student to verify that the parametric equations of this line are

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta,$$

where θ is a fixed number and r is the parameter.

Example 2. Derive parametric equations of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



It is left for the student to prove that

$$x = a\cos\theta, \quad y = b\sin\theta$$

are parametric equations of this ellipse, a and b being the semiaxes and θ the angle shown in the figure.

SUGGESTION: Note that if these equations hold, then $\dfrac{x^2}{a^2} = \cos^2\theta$ and $\dfrac{y^2}{b^2} = \sin^2\theta$.

The curve traced by a point on a circle which rolls along a straight line is called the **cycloid.**

Example 4. Find the equation of the cycloid traced by a point on a circle of radius r if it rolls along the x-axis.

Initially place the circle so that the point P is at the origin and let θ be the angle through which the radius of the circle turns.

In the figure the coördinates of P are $x = OQ$ and $y = QP$.

Then $OQ = OR - PT = r\theta - r\sin\theta$ and $QP = RC - TC = r - r\cos\theta$.

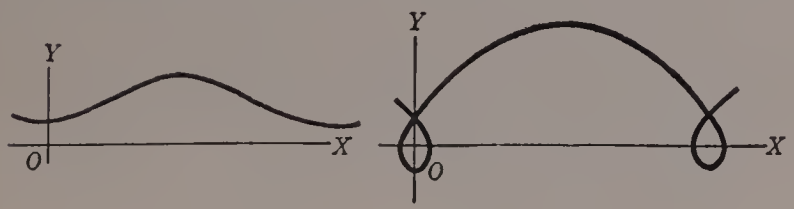
Hence $$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$$

are the required parametric equations.

EXERCISES

1. A circle is made to roll along the x-axis as in Example 4; find the equation of the path traced by a point on a fixed radius of the circle when the distance a from the center to the point is less than r.

2. Find the equation of the path traced as in Ex. 1 by a point which is on a radius extended; that is when a is greater than r.



The general shapes of the curves in Exs. 1 and 2 are shown in the figure.

These curves are called **trochoids.** Variations of these curves are obtained by letting the circle roll on the outside or inside of a fixed circle instead of along a straight line.

176. The folium of Descartes.—In some cases the locus of an equation is found more easily after a transformation of the axes. Such a case is the folium of Descartes whose equation is usually written in the form

$$x^3 + y^3 = 3axy.$$

Turning the axes through the angle $\frac{\pi}{4}$ by substituting

$$x = \frac{1}{2}\sqrt{2}(x' - y'), \quad y = \frac{1}{2}\sqrt{2}(x' + y')$$

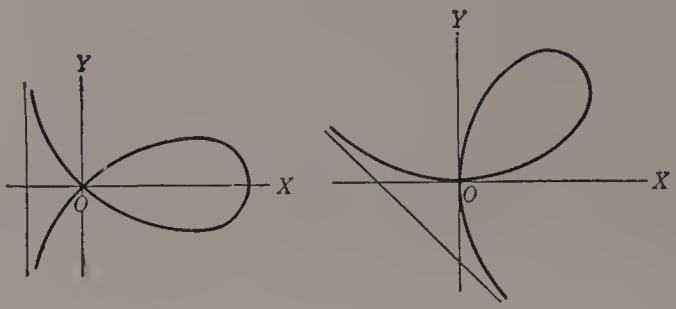
and suppressing primes we have,

$$\frac{\sqrt{2}}{4}(x - y)^3 + \frac{\sqrt{2}}{4}(x + y)^3 = \frac{3a}{2}(x - y)(x + y)$$

which reduces to

$$y^2 = x^2\frac{(3a - \sqrt{2}x)}{3(\sqrt{2}x + a)}.$$

Then, (a) the curve is symmetrical with respect to the x-axis, (b) $y = 0$ for $x = 0$ and for $x = \frac{3a}{\sqrt{2}}$, and (c) the curve approaches the line $x = -\frac{a}{\sqrt{2}}$ as an asymptote.



The figure to the left is the locus of $y^2 = x^2\frac{(3a - \sqrt{2}x)}{3(\sqrt{2}x + a)}$, and the one to the right is the locus of $x^3 + y^3 = 3axy$. In the second figure the line $x + y + a = 0$ is the asymptote.

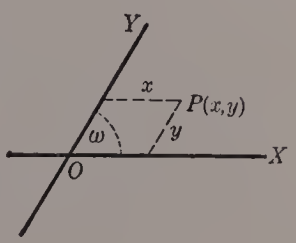
CHAPTER XV

SUPPLEMENTARY TOPICS AND PROBLEMS

This chapter contains topics, exercises, and problems which are supplementary to Chapters I to VIII.

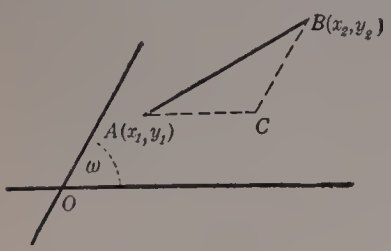
Supplementary Material for Chapters I and III

177. Oblique coördinates.—Oblique coördinates differ from rectangular coördinates only in that the axes are not at right angles to each other. The coördinates of a point are its distances from the two axes when each of these is measured along a line parallel to one of the axes. The Greek letter ω will be used to denote the angle formed by the positive parts of the axes.



Positive and negative coördinates are used exactly as in rectangular coördinates.

178. Distance between two points in oblique coördinates.—Let $A(x_1, y_1)$, $B(x_2, y_2)$ be any two points. Draw AC and BC parallel to the axes forming the triangle ABC. In this figure $\angle ACB$ is the supplement of ω and hence $\cos ACB = -\cos \omega$.



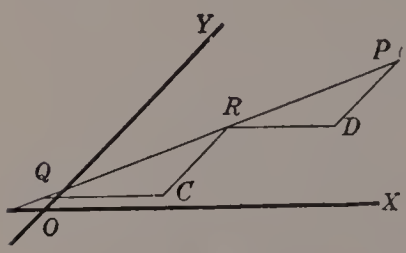
Then by trigonometry, (§253)

$$\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 - 2\overline{AC}\cdot\overline{CB}\cdot\cos ACB.$$

In this equation the absolute values of the segments are used. Obviously, $AC = x_2 - x_1$ and $CB = y_2 - y_1$, and hence

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 - 2(x_2 - x_1)(y_2 - y_1)\cos\omega} \quad [49]$$

179. Equation of straight line in oblique coördinates.—If in the figure the points $Q(x_1, y_1)$, $R(x_2, y_2)$, and $P(x, y)$ are in a straight line, then it follows exactly as in the case of rectangular coördinates that the triangles QCR and RDP are similar and hence



$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

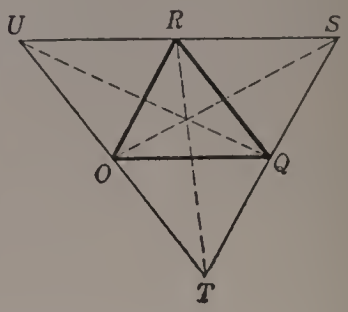
which is the equation of the straight line QR. Since this is a first degree equation in x and y and since any straight line is determined by two points, it follows that the equation of a straight line is of the general form $Ax + By + C = 0$.

The equations $x = a$, $y = b$ represent lines parallel to the coördinate axes.

It follows that problems on the intersections of straight lines may be solved in the same manner when the coördinates are oblique as when they are rectangular.

In some cases oblique coördinates have a distinct advantage over rectangular coördinates since the axes may be drawn so as to coincide with any two non-parallel lines.

Example. Lines are drawn through the vertices of a triangle parallel to the opposite sides of the triangle and the intersections of pairs of these lines are joined to the opposite vertices of the triangle. Show that these lines meet in a point.



SOLUTION: Let OQR be the given triangle and OQ and OR the coördinate axes. Let $OQ = a$ and $OR = b$. Show that (1) $x = a$, $y = b$, $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a} + \frac{y}{b} = 0$ are the equations of the lines TS, US, RQ, TU respectively; (2) the coördinates of the points S, T, U are $S(a, b)$, $T(a, -b)$, $U(-a, b)$; (3) the equations of the lines SO, TR, UQ are $bx - ay = 0$, $2bx + ay = ab$, $bx + 2ay = ab$. Then show that these lines meet in a point.

180. Point dividing a line segment in a given ratio.—By a study of §§10, 11 in Chapter I it will be found that the formulas

$$x = \frac{mx_2 + nx_1}{m+n}, \qquad y = \frac{my_2 + ny_1}{m+n}$$

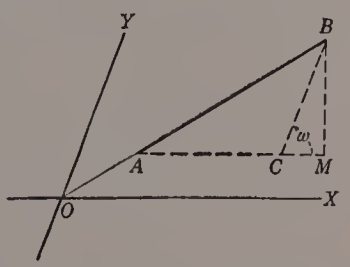
giving the coördinates of the point dividing the segment $A(x_1, y_1)$, $B(x_2, y_2)$ in the ratio $m : n$ are obtained by using the properties of similar triangles. Since the triangles remain similar when the axes are oblique these formulas hold in this case also.

EXERCISES

1. Prove that the medians of a triangle meet in a point which is two-thirds the distance from each vertex to the middle point of the opposite side. Compare solution of this problem suggested on page 14.

2. The points P, Q, R are on the sides of the triangle ABC so that $AP : PB = m : n$ and $BQ : QC = p : q$; find the ratio $AR : RC$ in terms of m, n, p, q, if BR passes through the intersection of CP and AQ. State an important special case.

181. Slope of line connecting two points.—Using oblique coördinates let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points. Construct lines AC and BC parallel to the axes forming the triangle ABC. Extend the side AC and draw BM perpendicular to it. The slope m of the line AB is the tangent of MAB or $\dfrac{MB}{AM}$.



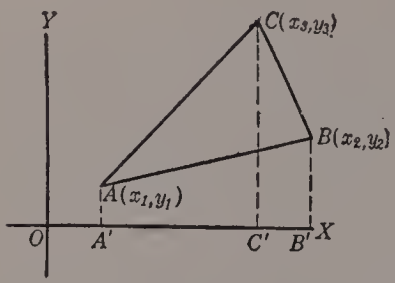
But $AC = x_2 - x_1, CB = y_2 - y_1, CM = CB\cos\omega = (y_2 - y_1)\cos\omega$, and $MB = (y_2 - y_1)\sin\omega$.

Hence
$$m = \frac{(y_2 - y_1)\sin\omega}{(x_2 - x_1) + (y_2 - y_1)\cos\omega} \qquad [50]$$

EXERCISE

Construct figures for the cases (1) when $x_2 > x_1$, and $y_2 < y_1$; (2) $x_2 < x_1$ and $y_2 > y_1$; (3) $x_2 < x_1$ and $y_2 < y_1$ and discuss the formula for each case. Assume that ω is acute.

182. The area of a triangle in rectangular coördinates.—The area of a triangle may be found in terms of the coördinates of its vertices.



Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of a triangle. Then, using rectangular coördinates, draw AA', BB', CC' perpendicular to the x-axis.

Obviously, in the figure as drawn,

Area ABC = Area $A'ACC'$ + Area $C'CBB'$ − Area $A'ABB'$

$= \frac{1}{2}(A'A + C'C) \cdot A'C' + \frac{1}{2}(C'C + B'B) \cdot C'B' - \frac{1}{2}(A'A + B'B) \cdot A'B'$

$= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$

$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$.

This expression may be positive or negative depending upon the order in which the vertices are arranged around the triangle (clock-wise or counter clock-wise), but its numerical value is always the area of the triangle.

Hence we write

$$\text{Area } ABC = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) \qquad [51]$$

This expression for the area may be written as at the right. (See §247)

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

PROBLEMS

1. Place the vertices of the triangle in various positions with respect to the coördinate axes and in different order around the triangle and show that the above formula represents the area in each case.

2. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ be the vertices of a convex quadrilateral. Show that the area is

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4).$$

SUGGESTION: Divide the quadrilateral into the two triangles ABC and ABD. Consider the case when the quadrilateral is not convex.

3. If the vertices of a convex polygon are (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n), show that its area is

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \ . \ . \ . \ + x_ny_1 - x_1y_n)$$

SUPPLEMENTARY PROBLEMS FOR CHAPTER I

1. Prove: In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the two diagonals plus four times the square of the segment connecting the middle points of the diagonals.

To what proposition does this reduce when the quadrilateral is a parallelogram?

Does this proposition hold when the quadrilateral is not convex?

2. (a) Find the coördinates of the middle point P_1 of the segment connecting $A(x_1, y_1)$ and $B(x_2, y_2)$.

(b) Find the coördinates of the point P_2 which divides the segment connecting P_1 and $C(x_3, y_3)$ in the ratio 2 : 1.

(c) Find the coördinates of the point P_3 which divides the segment connecting P_2 and $D(x_4, y_4)$ in the ratio 3 : 1.

(d) Continue in this manner, finally finding the coördinates of the point P_{n-1} which divides the segment connecting P_{n-2} and $N(x_n, y_n)$ in the ratio $(n-1) : 1$.

The point P_{n-1} obtained in this manner is called the **centroid** or **center of gravity** of the points $A, B, \ldots, N$.

3. Show that the centroid of the points $A, B, \ldots, N$ obtained in Ex. 2 is independent of the order in which the points are taken.

4. Show that the centroid of the vertices of a parallelogram is the middle point of either diagonal.

5. Show that the area of the triangle whose vertices are (a, b), (a^2, b^2), (a^3, b^3) is $ab(a-b)(a-1)(1-b)$.

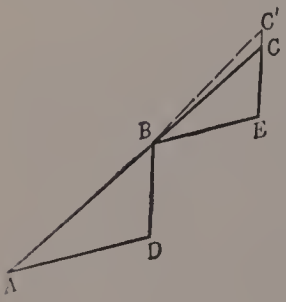
6. Find the area of the quadrilateral whose vertices are in order (a, b), (a^2, b^2), (a^3, b^3), (a^4, b^4).

7. Prove that the sum of the squares of two sides of a triangle is equal to twice the square of the median upon the third side plus twice the square of half the third side.

8. Find the lengths of the medians of the triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$. Arrange these lengths in order of magnitude if it is given that $AB < BC < CA$.

9. Show that the difference of the squares of two sides of a triangle is 3 times the difference of the squares of the medians upon these sides.

183. A theorem on similar triangles.—The following theorem is the basis for the proof that a first degree equation in two variables represents a straight line.



If two triangles ADB and BEC are similar and if AD is parallel to BE and DB is parallel to EC, then A, B, C lie on a straight line.

PROOF: Suppose C is not on the straight line AB, and that AB cuts the line EC in the point C'. Prove $EC = EC'$, and that hence C and C' coincide. The details are left for the student.

184. Locus of $Ax + By + C = 0$.—Using the theorem of §183 prove that if x_1, y_1; x_2, y_2; x_3, y_3 satisfy the equation $Ax + By + C = 0$, then the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear. Also prove that if x_1, y_1; x_2, y_2 satisfy the equation and x_3, y_3 do not, then the three points are not collinear.

Does your proof apply when the axes are oblique? why?

185. Determinant form of equation of straight line.—Prove that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is the equation of a straight line through the points (x_1, y_1), (x_2, y_2). (See §249.)

186. The homogeneous equation in two variables.—An equation is said to be **homogeneous** in x and y if in each term the sum of the exponents of the two variables is the same. Thus $Ax^2 + 2Hxy + By^2 = 0$ is a homogeneous equation of the second degree, and in general

$$A_0x^n + A_1x^{n-1}y + A_2x^{n-2}y^2 + \ldots + A_{n-1}xy^{n-1} + A_ny^n = 0$$

is a homogeneous equation of nth degree.

Consider first the second degree equation.

Dividing $Ax^2 + 2Hxy + By^2 = 0$ by y^2 and solving for x/y,

$$\frac{x}{y} = \frac{-H \pm \sqrt{H^2 - AB}}{A}.$$

Hence $Ax = y(-H + \sqrt{H^2 - AB})$ and $Ax = y(-H - \sqrt{H^2 - AB})$, which represent two straight lines through the origin provided $H^2 - AB$ is positive.

We now prove the general theorem:

If a homogeneous equation in two variables x *and* y *is reduced to an equation in* $\frac{x}{y}$ *and if this equation has* m *real roots, then the locus of the given equation is* m *straight lines through the origin.*

PROOF: By hypothesis

$$A_0\left(\frac{x}{y}\right)^n + A_1\left(\frac{x}{y}\right)^{n-1} + \ldots\ldots + A_{n-1}\frac{x}{y} + A_n = 0$$

has m real roots $r_1, r_2, \ldots, r_m$.

Then $$A_0\left(\frac{x}{y} - r_1\right)\left(\frac{x}{y} - r_2\right) \ldots\ldots \left(\frac{x}{y} - r_m\right)Q = 0,$$

and $x = r_1y$, $x = r_2y$, ... , $x = r_my$ all satisfy the given equation. But each of these represents a straight line through the origin. (See §246.)

Note that $A_0 \cdot Q$ is the quotient remaining after dividing the given polynomial by $\frac{x}{y} - r_1, \frac{x}{y} - r_2, \ldots\ldots\ldots$ By hypothesis there are no real values of x and y which make Q equal to zero. Hence the equation $Q = 0$ has no real locus and the m straight lines just found constitute the complete locus.

EXERCISES

1. Find the loci of the equations $x^3 - y^3 = 0$ and $x^3 + y^3 = 0$.
2. Find the locus of the equation $x^3 - 7xy^2 + 6y^3 = 0$.
3. Show that if $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ is the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), then it follows that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ is the equation of the line through the points (x_1, y_1), (x_2, y_2).

SUPPLEMENTARY PROBLEMS ON CHAPTER III

1. The ratio of the distances from P to the lines $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ is $m : n$; find the equation of the locus of P.

2. The perpendicular p from the origin upon a straight line makes the angles α and β with the coördinate axes. Show that $x \cos \alpha + y \cos \beta = p$ is the equation of the line.

3. Find the locus of the point P if the sum of its distances from two lines is a constant k.

4. Find the locus of the point P if the sum of its distances from n straight lines is a constant k.

5. Find the condition that the three lines $y = mx + b$, $y = m_1x + b_1$, $y = m_2x + b_2$ shall pass through the same point.

6. Find the equation of the line through the intersection points of the lines

$$\begin{cases} x - 2y - 6 = 0 \\ 2x + 3y + 2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} 5x + y + 4 = 0 \\ 2x - 7y - 6 = 0 \end{cases}$$

without first finding the intersection points of the lines.

SUGGESTION: Find k and k' so that the lines $x - 2y - 6 + k^1(2x + 3y + 2) = 0$ and $5x + y + 4 + k\,(2x - 7y - 6) = 0$ shall be identical.

7. Find the area of the triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ by the following steps: (a) find the length of the segment BC, (b) find the equation of the line BC, (c) find the distance from A to the line BC.

8. Prove that the perpendicular bisectors of the sides of a triangle meet in a point.

9. Prove that if two medians of a triangle are equal, the triangle is isosceles.

10. Prove that the bisectors of the angles of a triangle meet in a point.

11. A trapezoid is formed by drawing lines parallel to the base of a triangle. Find the locus of the intersection point of the diagonals of such trapezoids.

12. A rectangle is constructed with one side on a side of a triangle and two vertices on the other two sides of the triangle; find the locus of the intersection point of the diagonals of such rectangles.

13. Prove that the bisector of an angle of a triangle divides the opposite side in a ratio which is equal to the ratio of the other two sides of the triangle.

14. The sides of a triangle are (1) $5x - 5y + 7 = 0$, (2) $2x + 3y - 8 = 0$, and (3) $x - 2y = 5$. Show that the bisector of the interior angle formed by (1) and (2) and the bisectors of the angles (interior or exterior) formed by (2) and (3), and by (3) and (1) all meet in a point.

15. State and prove a general theorem of which the preceding is a special case.

16. Show that the bisectors of the exterior angles of a triangle meet the opposite sides, extended, in collinear points.

17. Find the equation of the locus of a point if the product of its distances from two given straight lines is k times the product of its distances from two other lines. What is the degree of the equation of this locus? (See Historical Sketch.)

Supplementary Material for Chapter IV

187. Determinant form of equation of circle.—The equation of a circle through the points (x_1, y_1), (x_2, y_2), (x_3, y_3) may be written

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \qquad (1)$$

To prove that this is the equation of the required circle show (a) the equation reduces to the general form

$$Ax^2 + Ay^2 + 2Dx + 2Ey + C = 0$$

where A, D, E, C are combinations of the given constants. Developing this determinant according to minors of the elements of the first row, we have

$$x^2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + y^2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - x\begin{vmatrix} x_1^2 + y_1^2 & y_1 & 1 \\ x_2^2 + y_2^2 & y_2 & 1 \\ x_3^2 + y_3^2 & y_3 & 1 \end{vmatrix}$$

$$+ y\begin{vmatrix} x_1^2 + y_1^2 & x_1 & 1 \\ x_2^2 + y_2^2 & x_2 & 1 \\ x_3^2 + y_3^2 & x_3 & 1 \end{vmatrix} - \begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 \\ x_2^2 + y_2^2 & x_2 & y_2 \\ x_3^2 + y_3^2 & x_3 & y_3 \end{vmatrix} = 0.$$

(b) Show that (1) is satisfied by x_1, y_1; x_2, y_2; x_3, y_3. (See §§248, 249).

EXERCISES

1. Using the determinant form write the equations of circles through the points: (1, 0), (4, 2), (0, 5); $(a, 0)$, $(0, b)$, (a, b); $(-a, 0)$, $(b, 0)$, $(2a, -b)$; (a, c), $(2a, -c)$, (c, a).

2. Study the case when the three points through which the circle is to pass lie on a straight line. How will that affect the determinant in equation (1) on page 209?

188. Radical axis of two circles.—Let

$$x^2 + y^2 + 2D_1x + 2E_1y + C_1 = 0 \qquad (1)$$
$$x^2 + y^2 + 2D_2x + 2E_2y + C_2 = 0 \qquad (2)$$

be the equations of any two circles. Subtracting members of these equations, we have

$$2(D_1 - D_2)x + 2(E_1 - E_2)y + C_1 - C_2 = 0. \qquad (3)$$

This straight line is the radical axis of circles (1) and (2).

It is shown in §55 that if (1) and (2) intersect, (3) passes through the points of intersection, and that if (1) and (2) are tangent to each other then (3) is tangent to both circles at their point of tangency.

We will now prove the following theorem:

The radical axis of two circles is the locus of the points from which tangents to the two circles are equal.

PROOF: Let

$$x^2 + y^2 + 2D_1x + 2E_1y + C_1 = 0 \qquad (1)$$
$$x^2 + y^2 + 2D_2x + 2E_2y + C_2 = 0 \qquad (2)$$

be the circles. Then

$$2(D_1 - D_2)x + 2(E_1 - E_2)y + C_1 - C_2 = 0 \qquad (3)$$

is the radical axis. Let $P(x_1, y_1)$ be a point on (3) from which tangents are drawn. Then by §53, the squares of the tangents are

$$x_1^2 + y_1^2 + 2D_1x_1 + 2E_1y_1 + C_1$$

and

$$x_1^2 + y_1^2 + 2D_2x_1 + 2E_2y_1 + C_2.$$

Using (3) these are easily proved equal.

In case the circles intersect the locus consists of only that part of the line which does not lie inside the circles.

For brevity the equation of a circle is often written $S = 0$. Then the equation of the radical axis of $S_1 = 0$, $S_2 = 0$ is $S_1 - S_2 = 0$.

EXERCISES

1. Find the ratio in which the radical axis of two circles cuts the line of centers.

2. Prove that for any three circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, the radical axes of the three pairs of circles $S_1 = 0$, $S_2 = 0$; $S_2 = 0$, $S_3 = 0$; $S_3 = 0$, $S_1 = 0$, all meet in a point.

This point is called the **radical center** of the circles.

3. Find the radical center of $x^2 + y^2 - 4x + 6y - 5 = 0$, $x^2 + y^2 + 7x - 5y + 12 = 0$, $x^2 + y^2 - 9x + 8y - 4 = 0$.

4. Find the point from which the tangents to the three circles $(x - 2)^2 + (y - 4)^2 - 16 = 0$, $(x + 4)^2 + (y + 6)^2 - 9 = 0$, and $(x - 8)^2 + (y - 10)^2 - 25 = 0$ are equal. Is there more than one such point?

5. Find the equation of a circle $S_1 = 0$ whose center is the radical center of the circles in Ex. 4, and whose radius is the length of a tangent from this point to one of the circles. Suppose that this circle meets one of the given circles $S_2 = 0$ in the point A. Show that the tangent to $S_2 = 0$ at A passes through the center of $S_1 = 0$.

6. Study the case when the radical center of three circles lies within at least one of them. (a) In this case must each circle cut each of the other two? (b) If the radical center lies within one of the circles must it lie within all of them? (c) If the radical center (x_1, y_1) lies within the circles, discuss the expression $x_1^2 + y_1^2 + 2Dx_1 + 2Ey_1 + C$ giving the square of the length of the tangent to one of the circles. How do these expressions compare for the three circles?

PROBLEMS

1. Find the equation of a circle whose center is on the line $2x - y = 10$ and which is tangent to the two lines $x - 5y = 12$ and $4x + y = 8$.

2. Find the equation of the circle whose diameter is the common chord of $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = r^2$.

3. Find the length of the chord cut off by $x^2 + (y - b)^2 = r^2$ on the line $\frac{x}{a} + \frac{y}{b} = 1$.

4. The line $x \cos \alpha + y \sin \alpha = p$ is tangent to a circle with its center at the origin. Find the coördinates of the point of contact.

5. Prove that $ax + by + a^2 + b^2 = 0$ is tangent to $x^2 + y^2 + ax + by = 0$ and find the point of contact.

6. Find the equation of a circle with center (a, a) which cuts a chord of length b from each axis.

7. From a fixed point P on a circle chords are drawn. Find the locus of their middle points.

SUGGESTION: In problems involving "any circle" and any point on it consider whether the equation of the circle may be written $x^2 + y^2 = r^2$ and a special point on it such as $(-r, 0)$ may be selected.

8. From a fixed point A on a circle secants AB are drawn. Find the locus of points P which divide each chord AB in the ratio $m : n$. Consider the cases when $m : n$ is positive and also when $m : n$ is negative.

9. Find the locus of P when $AP : BP = m : n$, A and B being fixed points.

10. The points A and B are fixed and C moves so that $\angle ACB$ is constant; find the locus of C.

11. Find the condition that the two circles $x^2 + y^2 + 2D_1x + 2E_1y + C_1 = 0$ and $x^2 + y^2 + 2D_2x + 2E_2y + C_2 = 0$ are tangent to each other.

12. Find the condition that the line $\frac{x}{a} + \frac{y}{b} = 1$ shall be tangent to the circle $x^2 + y^2 = r^2$.

13. Find a point P from which the tangents to the three circles $x^2 + y^2 + 4x - 10y + 2 = 0$, $x^2 + y^2 + 2x + 2y - 6 = 0$, and $x^2 + y^2 + 8x + 4y - 12 = 0$ are equal.

14. Show analytically that the perpendicular from any point on a circle to a chord is a mean proportional between the perpendiculars from the same point on the tangents at the extremities of the chord.

15. Given the circle $x^2 + y^2 + 2Dx + 2Ey + C = 0$ (1), and the line $Ax + By + C_1 = 0$ (2), show that the circles in the system $x^2 + y^2 + 2Dx + 2Ey + C + k(Ax + By + C_1) = 0$ have their centers on the line through the center of (1) which is perpendicular to (2).

16. Prove that in two circles the square of the tangent to one circle from a point P on the other is proportional to the distance from P to the radical axis of the two circles.

17. Given the circles $x^2 + y^2 - 10x - 6y + 4 = 0$ (1) and $x^2 + y^2 + 8x - 6 = 0$ (2). Find a point P such that the tangents from P to (1) form

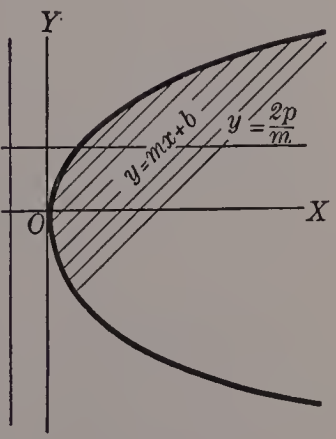
a right angle with each other and such that the tangents from P to (2) make an angle of 60° with each other.

Supplementary Material for Chapter V

189. Diameters of the parabola.—Consider the parabola $y^2 = 4px$ and the secant line $y = mx + b$.

Solving simultaneously for y

$$y = \frac{2p \pm 2\sqrt{p^2 - mpb}}{m}.$$



Hence the y-coördinate of the middle point of the chord is $\frac{2p}{m}$. That is, the middle points of all chords with slope m lie on the line $y = \frac{2p}{m}$. The locus of the middle points of a set of parallel chords is called a diameter of the parabola. It follows that every line parallel to the axis of the parabola is a diameter.

By solving we find that the tangent $y = mx + \frac{p}{m}$, which is parallel to the chords bisected by the diameter $y = \frac{2p}{m}$ meets the parabola at the point for which $y = \frac{2p}{m}$.

Hence we have:

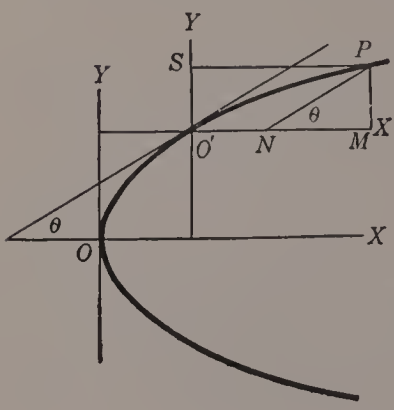
The tangent to the parabola at the extremity of a diameter is parallel to the chords bisected by the diameter. Compare §§139, 141.

EXERCISES

1. Show that the middle points of any two parallel chords of a parabola are equally distant from its axis.

2. Find the slope of the chords bisected by the line $y = b$, (a) when b is positive, (b) when b is negative.

190. Parabola referred to a diameter and corresponding tangent as axes.—Let m be the slope of the tangent to the parabola $y^2 = 4px$ at a point O'. Then, using results obtained in §189, the coördinates of O' are $x = \frac{p}{m^2}$, $y = \frac{2p}{m}$. Transferring to rectangular coördinates with O' as origin and axes parallel to the original axes and suppressing primes, we have



$$y^2 + \frac{4p}{m}y = 4px. \tag{1}$$

The coördinates of a point P in the new rectangular coördinates are $x' = O'M$, $y' = MP$. Using the diameter and tangent as axes the coördinates are $x' = O'N$, $y' = NP$.

Denoting the angle formed by the diameter and tangent by θ and noting that $\tan\theta = m$,

$$MP = NP\sin\theta,\ SP = O'N + NM = O'N + NP\cos\theta.$$

Hence $$y = y'\sin\theta,\ x = x' + y'\cos\theta.$$

Substituting and again suppressing primes we have

$$y^2\sin^2\theta + \frac{4p}{m}y\sin\theta = 4px + 4py\cos\theta.$$

Since $\frac{\sin\theta}{m} = \cos\theta$, this reduces to $y^2 = \frac{4px}{\sin^2\theta}$.

We now find that the distance from the focus $(p, 0)$ to the new origin $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$ is $p\frac{m^2+1}{m^2} = \frac{p}{\sin^2\theta}$.

Hence, if p' is the distance from the focus to the new origin, then $$y^2 = 4p'x$$ is the equation of the parabola.

Since the y-axis is tangent to the parabola at its vertex we have the general theorem:

If the diameter and the tangent of a parabola meeting it at O′ *are used as axes and if* p′ *is the distance from the focus to* O′, *then the equation of the parabola is* $y^2 = 4p'x$.

PROBLEMS

1. Prove that $y = mx - mp(2 + m^2)$ is the equation of the normal with slope m to the parabola $y^2 = 4px$.

2. Using the equation of the normal given in Problem 1, show that from an exterior point (x_1, y_1) there is always at least one normal, and there may be two or three.

SUGGESTION: The cubic equation in m, $y_1 = mx_1 - pm(2 + m^2)$ always has one real root, may have three distinct real roots, or one single and two coincident roots.

3. Two straight lines are drawn through the vertex of a parabola at right angles to each other meeting the curve in A and B. Show that AB meets the axis in a fixed point.

4. From a point on the latus rectum of a parabola perpendiculars are drawn to the tangents at the extremities. Prove that the line joining the feet of these perpendiculars is a tangent to the parabola.

5. If y_1, y_2, y_3 are the ordinates of three points on a parabola $y^2 = 4px$, prove that the area of the triangle determined by these points is

$$\frac{1}{8p}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1).$$

6. The normal to a parabola $y^2 = 4px$ at $P(x_1, y_1)$ meets the x-axis in the point $P(x_2, 0)$. Prove that $x_2 - x_1 = p$.

7. Prove that normals to a parabola which are perpendicular to each other intersect on the parabola $y^2 = px - 3p^2$.

8. Prove that the locus of the point of intersection of two tangents to a parabola is a straight line if the product of their slopes is constant.

9. If A, B, and C are three points on the parabola $y^2 = 4px$, the ordinates of which are in geometrical progression, show that the tangents at A and B will meet on the ordinate of C.

10. If a straight line be drawn from the origin to any point Q of the line $y = a$, and if a point P be taken on this line such that its ordinate is equal to the abscissa of Q, show that the locus of P is a parabola.

11. The normal at A, the upper end of the latus rectum, meets the parabola again at B. Show that the diameter on which the tangents at A and B intersect passes through the other extremity of the latus rectum.

12. Prove that the locus of the middle points of all chords of a parabola which pass through a fixed point on it is a parabola.

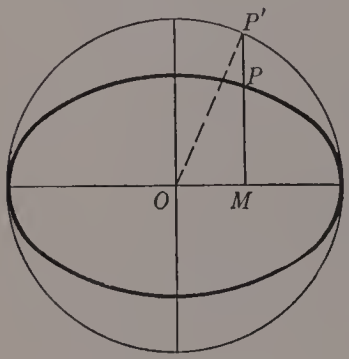
13. Two equal parabolas have their axes parallel and have a common tangent at their vertices; straight lines are drawn parallel to the axes. Show that the locus of the middle points of the parts of the lines intercepted between the curves is a parabola equal to the given parabolas.

14. The base of a triangle is $2b$ and the sum of the tangents of the angles at the base is s. Find the locus of the vertex. (Take the base of the triangle as the axis of x and the middle point of the base as origin.)

15. Prove: If tangents to a parabola are drawn from the extremities of a focal chord, then (a) the tangents intersect on the directrix, (b) the straight line drawn from the point of intersection of tangents to the focus is perpendicular to the chord.

Supplementary Material For Chapter VI

191. The auxiliary circle of an ellipse.—The circle constructed on the major axis of an ellipse as a diameter is called the **auxiliary circle** of the ellipse. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,(a>b)$ the equation of this circle is



$$x^2 + y^2 = a^2.$$

If the ordinate of any point P of the ellipse be extended to meet the auxiliary circle at P' then

$$MP = \frac{b}{a}\sqrt{a^2 - x^2} \text{ and } MP' = \sqrt{a^2 - x^2}.$$

Hence $MP' : MP = a : b$.

This property of the ellipse shows that a circle seen from any point which is not in the perpendicular drawn to its plane at the center appears in the form of a true ellipse, the eccentricity of the ellipse depending on the angle at which the circle is seen.

192. The eccentric angle.—If through a point P the ordinate is drawn meeting the x-axis in M and the auxiliary circle in P' then the angle MOP' is called the **eccentric angle** of the point P.

193. Perpendicular from focus meets tangent on auxiliary circle.—Let $y = mx + \sqrt{a^2m^2 + b^2}$ be the equation of a tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Let $y = -\frac{1}{m}(x - ae)$ be the equation of a line through one of the foci perpendicular to the tangent.

Squaring both sides in the equations

$$y - mx = \sqrt{a^2m^2 + b^2} \text{ and } my + x = ae$$

and adding, $y^2 + m^2x^2 + m^2y^2 + x^2 = a^2e^2 + a^2m^2 + b^2$

or, $x^2(1 + m^2) + y^2(1 + m^2) = a^2m^2 + b^2 + a^2e^2 = a^2(1 + m^2)$.

(Note that $a^2e^2 = a^2 - b^2$.)

Hence $x^2 + y^2 = a^2$ is the result obtained by eliminating m from the equations $y - mx = \sqrt{a^2m^2 + b^2}$ and $my + x = ae$, and the lines meet on the auxiliary circle.

Since the auxiliary circle meets the ellipse in only two points, it follows that a focal radius is perpendicular to the tangents only at the extremities of the major axis.

194. The director circle.—The circle $x^2 + y^2 = a^2 + b^2$ is called the **director circle** of the ellipse.

In §80, Example 1, the following theorem was proved:

Two perpendicular tangents to an ellipse meet on the director circle.

EXERCISES

1. Prove the propositions of §193 by taking the line through $(-ae, 0)$.

2. By means of §191 construct a series of points which shall be on an ellipse: (a) by making each ordinate ¼ the corresponding ordinate of the auxiliary circle; (b) by making each ordinate ¾ the ordinate of the circle.

3. On the ordinates MP' of a circle $x^2 + y^2 = a^2$ the point P is located so that $MP : MP' = m : n$; find the locus of P.

4. Prove a proposition like the one in §194 for the case when $a = b$, that is, for a circle.

PROBLEMS

1. A circle described on a focal radius of an ellipse as a diameter is tangent to the auxiliary circle.

2. Prove that the normals to an ellipse and to its auxiliary circle at corresponding points (that is, at points which have the same abscissa) meet on the circle $x^2 + y^2 = a^2 + b^2$.

3. Two conjugate diameters of an ellipse are drawn, and their four extremities are joined to any point on a given circle whose center is at the center of the ellipse. Show that the sum of the squares of these four lines is constant.

4. In the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ find the point for which the eccentric angle is 30°.

5. The point P moves along an ellipse whose vertices are A' and A. A line is drawn from A perpendicular to PA' and another from A' perpendicular to PA. Find the locus of the point of intersection of these lines.

6. A line parallel to the minor axis of an ellipse is drawn midway between a focus and the corresponding directrix. Show that the product of the perpendiculars on this line from the extremities of a focal chord is constant.

7. From a point P on an ellipse lines PF' and PF are drawn through the foci, meeting the corresponding directrices in Q and R. Show that the locus of the point of intersection of QF and RF' is an ellipse.

8. Two equal circles are tangent to each other. Find the locus of a point which moves so that the sum of the tangents from it to the two circles is constant.

9. A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that the locus is an ellipse, and find its eccentricity in terms of the angle between the lines.

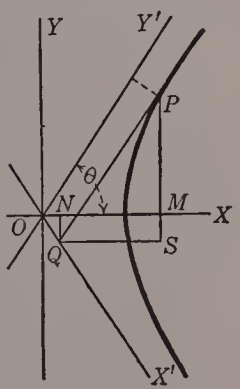
10. Two tangents to an ellipse are so drawn that the product of their slopes is constant. Show that the locus of their point of intersection is an ellipse or hyperbola according as the product is negative or positive.

11. Through a point P on an ellipse draw chords PR and PQ and through R and Q draw chords PS and PT parallel respectively to PQ and PR. Prove that ST is parallel to the tangent at P.

12. Tangents are drawn to the two ellipses $b_1^2x^2 + a^2y^2 = a^2b_1^2$, and $b_2^2x^2 + a^2y^2 = a^2b_2^2$ at the points (x_1, y_1) and (x_1, y_2). Prove that these tangents meet the x-axis in the same point. Show how this result may be used in constructing a tangent to an ellipse.

Supplementary Material For Chapter VII

195. The hyperbola referred to its asymptotes as axes.— In the figure the asymptotes of the hyperbola are marked as the new axes. Let $P(x, y)$ be any point on the hyperbola and x', y' its coördinates when referred to the new axis. The angle made by the old x-axis and an asymptote is denoted by θ.



Then $x = OM = ON + NM$

$= x'\cos\theta + y'\cos\theta = (x' + y')\cos\theta$

$y = SP - SM = y'\sin\theta - x'\sin\theta$

$= (y' - x')\sin\theta.$

Substituting in $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and suppressing primes, we have

$$\frac{\cos^2\theta(x + y)^2}{a^2} - \frac{\sin^2\theta(x - y)^2}{b^2} = 1. \tag{1}$$

But $\tan\theta = \dfrac{b}{a}$. Hence $\sin\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$ and $\cos\theta = \dfrac{a}{\sqrt{a^2 + b^2}}$.

Substituting these values in (1) we have

$$4xy = a^2 + b^2, \tag{2}$$

which is the equation of the hyperbola when referred to its asymptotes as axes.

Compare this result with that obtained in §98 for the case of the rectangular hyperbola.

PROBLEMS

1. Show that the focal distance of any point on a hyperbola is equal to the distance from the point to the corresponding directrix along a line parallel to an asymptote.

2. Prove that a circle with a focus of a hyperbola as center and with radius equal to half the conjugate axis, touches the asymptotes in the points where they intersect the directrix.

3. If two concentric equilateral hyperbolas are constructed, the axes of one being the asymptotes of the other, show that they will intersect at right angles. (Two curves intersect at right angles if their tangents at the point of intersection are at right angles to each other.)

4. Prove that the equation of the tangent to the hyperbola, referred to its asymptotes as axes, is $x_1y + y_1x = \frac{a^2 + b^2}{2}$, where (x_1, y_1) is the point of contact.

5. Find the locus of the intersection of a pair of perpendicular tangents to a hyperbola.

6. Find the locus of the foot of the perpendicular from either focus of a hyperbola to any tangent.

7. The base c of a triangle is fixed, and the opposite vertex C moves so that one base angle is twice the other; find the locus of C.

8. Prove that the part of a normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ which lies between the axes is divided by the curve into segments which are in the ratio $a^2 : b^2$.

9. Let H and H' represent two conjugate hyperbolas. Let the tangent to H at its vertex meet H' in P. Prove that the tangent to H' at P meets H in its other vertex.

10. Prove that the area of the triangle formed by the asymptotes and a tangent to a hyperbola is constant.

12. The ordinate NP of a hyperbola is produced to Q so that $NQ = FP$ (where F is a focus); find the locus of Q.

13. A line parallel to the conjugate axis is drawn midway between a focus and the corresponding directrix of a hyperbola. Show that the product of the perpendiculars on this line from the extremities of a focal chord is constant.

196. Conic through five points.—The general equation of the conic is

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0. \qquad (1)$$

Dividing by one of the coefficients, A for instance, leaves five coefficients to be determined, and hence the equation is satisfied by five sets of values of x and y.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), be any five points. Then the equation,

$$\begin{vmatrix} x^2 & y^2 & xy & x & y & 1 \\ x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3y_3 & x_3 & y_3 & 1 \\ x_4^2 & y_4^2 & x_4y_4 & x_4 & y_4 & 1 \\ x_5^2 & y_5^2 & x_5y_5 & x_5 & y_5 & 1 \end{vmatrix} = 0 \tag{2}$$

is satisfied by each of these pairs of coördinates since the substitution of any one pair, x_1, y_1 for instance, gives two identical rows. Also equation (2) is of the general form (1) since by expanding we have

$$\begin{aligned} x^2 & \begin{vmatrix} y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ y_3^2 & x_3y_3 & x_3 & y_3 & 1 \\ y_4^2 & x_4y_4 & x_4 & y_4 & 1 \\ y_5^2 & x_5y_5 & x_5 & y_5 & 1 \end{vmatrix} - y^2 \begin{vmatrix} x_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & x_5 & y_5 & 1 \end{vmatrix} + xy \begin{vmatrix} x_1^2 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} \\ - x & \begin{vmatrix} x_1^2 & y_1^2 & x_1y_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3y_3 & y_3 & 1 \\ x_4^2 & y_4^2 & x_4y_4 & y_4 & 1 \\ x_5^2 & y_5^2 & x_5y_5 & y_5 & 1 \end{vmatrix} + y \begin{vmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & 1 \\ x_3^2 & y_3^2 & x_3y_3 & x_3 & 1 \\ x_4^2 & y_4^2 & x_4y_4 & x_4 & 1 \\ x_5^2 & y_5^2 & x_5y_5 & x_5 & 1 \end{vmatrix} - \begin{vmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 \\ x_3^2 & y_3^2 & x_3y_3 & x_3 & y_3 \\ x_4^2 & y_4^2 & x_4y_4 & x_4 & y_4 \\ x_5^2 & y_5^2 & x_5y_5 & x_5 & y_5 \end{vmatrix} = 0. \end{aligned}$$

197. Confocal ellipses and hyperbolas.—By §70 the foci of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are $(\pm\sqrt{a^2 - b^2}, 0)$ and by §88 the foci of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ are $(\pm\sqrt{a^2 + b^2}, 0)$.

Consider the ellipse

$$\frac{x^2}{a^2 - k} + \frac{y^2}{b^2 - k} = 1 \tag{1}$$

where $b < a$ and $0 \leqq k < b^2$. Obviously for every such value of k, (1) represents an ellipse whose foci are $(\pm\sqrt{a^2 - b^2}, 0)$. That is, (1) represents a set of confocal ellipses.

If in (1) $b^2 < k < a^2$, the equation may be written

$$\frac{x^2}{a^2 - k} - \frac{y^2}{k - b^2} = 1. \tag{2}$$

For every such value of k (2) represents a hyperbola, whose foci are $(\pm\sqrt{a^2 + b^2}, 0)$.

Example. Show that in the system

$$\frac{x^2}{a^2 - k} + \frac{y^2}{b^2 - k} = 1$$

each ellipse is perpendicular to every hyperbola.

SOLUTION: Let (x_1, y_1) be a point on the ellipse

$$\frac{x^2}{a^2 - k_1} + \frac{y^2}{b^2 - k_1} = 1. \tag{1}$$

Then by §76

$$\frac{x_1 x}{a^2 - k_1} + \frac{y_1 y}{b^2 - k_1} = 1$$

is tangent to the ellipse, and the slope of this line is $-\dfrac{x_1}{y_1} \cdot \dfrac{b^2 - k_1}{a^2 - k_1}$. The slope of the tangent to the hyperbola,

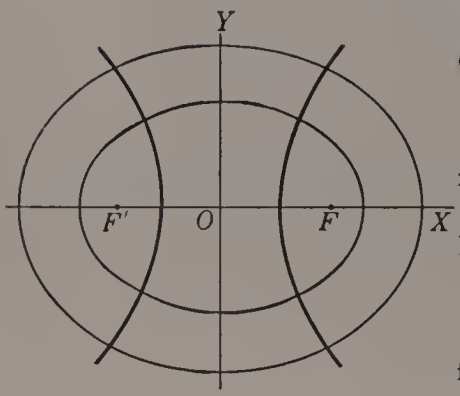
$$\frac{x^2}{a^2 - k_2} - \frac{y^2}{k_2 - b^2} = 1 \tag{2}$$

at the point (x_1, y_1) is $\dfrac{x_1}{y_1} \cdot \dfrac{k_2 - b^2}{a^2 - k_2}$.

The product of these slopes is $-\dfrac{x_1^2}{y_1^2} \cdot \dfrac{(k_2 - b^2)(b^2 - k_1)}{(a^2 - k_2)(a^2 - k_1)}$.

Since (x_1, y_1) is on (1) and also on (2) this reduces to -1 and the curves are normal to each other at this point.

EXERCISES



1. Construct on the same axes the ellipses

$$\frac{x^2}{16 - k} + \frac{y^2}{9 - k} = 1$$

for $k = 0$, $k = 2$, $k = 4$, $k = 6$, $k = 8$.

2. Construct on the same axes the hyperbolas

$$\frac{x^2}{16 - k} + \frac{y^2}{9 - k} = 1$$

for $k = 10$, $k = 12$, $k = 14$.

3. Discuss the change in the locus of $\dfrac{x^2}{16 - k} + \dfrac{y^2}{9 - k} = 1$, (a) as k changes from 0 to 9, (b) as k changes from 9 to 16.

198. The parabola as the limiting case of the ellipse and the hyperbola.—Consider the equation

$$Ax^2 + By^2 + 2Dx + 2Ey + C = 0. \tag{1}$$

Let A be a fixed positive number and let C be negative.

Then $BD^2 + AE^2 - ABC = K$ is positive when B is positive (§103).

Hence under these conditions ($A > 0$ and $C < 0$) (1) represents a real locus for all values of B. (§106).

Then, (a) for B positive the locus of (1) is an ellipse;
(b) for $B = 0$ the locus is a parabola;
(c) for B negative the locus is a hyperbola.

From this we may conclude that for a series of positive values $B_1, B_2, B_3, \ldots, B_n, \ldots$ decreasing toward zero, we have a series of ellipses which approach the form of a parabola as a limit, and for a series of negative values of B which approach zero as a limit we have a series of hyperbolas which approach the form of a parabola as a limit.

The following examples illustrate the parabola as the limit of a series of ellipses and hyperbolas.

Example 1. Find the limit of the ellipse $\dfrac{(x-a)^2}{a^2} + \dfrac{y^2}{p(2a-p)} = 1$ as a increases indefinitely.

SOLUTION: From the equation of the given ellipse

$$y^2 = 4px + \frac{p^2x^2}{a^2} - \frac{2p^2x}{a} - \frac{2px^2}{a}.$$

Hence for any fixed value of x, y^2 approaches $4px$ as a limit as a becomes indefinitely great.

Note, (a) that the given ellipse has its left vertex at the origin; (b) that $b^2 = p(2a - p)$ is such that the distance from the origin to the focus is $a - ae = a - \sqrt{a^2 - b^2} = a - \sqrt{a^2 - 2ap + p^2} = p$. Hence for all values of a the ellipse has a fixed vertex and a fixed focus. As a grows large the center of the ellipse and its right focus and vertex recede indefinitely toward the right, and the left part of the ellipse approaches more and more nearly to the parabola $y^2 = 4px$.

Example 2. Find the limit of the hyperbola $\frac{(x+a)^2}{a^2} - \frac{y^2}{p(2a+p)} = 1$ as a increases indefinitely.

SOLUTION: The limiting curve of this set of hyperbolas is $y^2 = 4px$. The proof is closely analogous to that given under Example 1, and is left to the student.

Show that for all values of a one focus of this set of hyperbolas remains fixed. How does the left branch of this hyperbola change as a increases?

199. Two straight lines as the limiting case of the ellipse and the hyperbola.—Consider the equation

$$Ax^2 + By^2 = C,$$

where A and C are both positive and fixed in value. If B is positive, the locus is an ellipse with $\sqrt{\frac{C}{A}}$ and $\sqrt{\frac{C}{B}}$ as semiaxes. If B decreases toward zero, then the axis of the ellipse which lies on the y-axis increases indefinitely while the axis which lies on the x-axis remains fixed. Hence the curve approaches the two straight lines $x = -\sqrt{\frac{C}{A}}$ and $x = \sqrt{\frac{C}{A}}$.

If B is negative the locus is a hyperbola which meets the x-axis in the points $\left(\pm\sqrt{\frac{C}{A}}, 0\right)$. As B decreases in absolute value toward zero the curve approaches the two straight lines $x = \pm\sqrt{\frac{C}{A}}$.

EXERCISES

1. Describe the change in the locus of $\frac{x^2}{a^2} + a^2y^2 = 1$ as a increases indefinitely.

2. Describe the change in the locus of $\frac{x^2}{a^2} - \frac{y^2}{a^2k^2} = 1$ as a decreases toward zero, k being a constant

CHAPTER XVI

CUMULATIVE REVIEWS

1

AFTER PAGE 26

1. Find the length of the segment whose end-points are $A(-7, 8)$ and $B(2, -1)$.

2. The line $x + y = 4$ meets the curve $y^2 = 8x$ in the points A and B; find the projections of AB on the two axes.

3. Find the projections on the two axes of the segment whose end-points are $(a + \sqrt{b}, \sqrt{a} + b)$ and $(\sqrt{b} - a, b - \sqrt{a})$.

4. Find the point A if $AP : PB = \frac{5}{9}$ and if the points P and B are $(7, 9)$ and $(-4, 5)$, the points A, P, B being collinear.

5. Discuss the locus $5x^2 - 9y^2 = 10$ as to symmetry.

6. Find the coördinates of P if $AP : PB = -\frac{3}{7}$ and if the points A and B are $(4, -6)$ and $(-3, 5)$, the points A, P, B being collinear.

2

AFTER PAGE 33

1. Find the sum of the projections on the x-axis of the segments AB and BC if the points are: $A(4, a)$, $B(c, \sqrt{b})$, $C(8, \sqrt{a})$. Also find the sum of the projections of these segments on the y-axis.

2. Find the coördinates of P if A, B, and P are collinear and if $AP : PB = -\frac{7}{3}$, the points A and B being $(3, -6)$ and $(-7, 4)$.

3. Discuss the locus of $8x^2 + 12y - 9 = 0$ as to symmetry.

4. The line $3x - y = 8$ meets the locus of $2x - 3y^2 + 8 = 0$ in the points P and Q; find the projections of the segment PQ on the two axes.

5. Write the equation of the line through the points $(7, 0)$ and $(0, 5)$. What is the slope of this line?

6. Discuss the lines $y = 3x + b$ for different values of b.

3

AFTER PAGE 33

1. Find the coördinates of the points P and P' if A, B, P, and P' are collinear and if $AP : PB = \frac{3}{4}$ and $AP' : P'B = -\frac{4}{3}$, the points A and B being $(-3, 8)$ and $(7, -2)$.

2. Find the equation of the line through the points $(-2, 3)$ and $(4, 6)$.

3. Write the equation of the line through the points $(3, 0)$ and $(0, -6)$. Reduce this equation to the form $y = mx + b$.

4. Find the slope of the line through the points $(5, -7)$ and $(-2, 6)$.

5. Under what conditions are the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ parallel?

6. A line through the point $(7, -4)$ has a slope $-\frac{2}{3}$; write its equation.

4

AFTER PAGE 35

1. Find the coördinates of P if it divides the segment connecting $A(-4, -7)$ and $B(6, 4)$ so that $AP : PB = -\frac{5}{8}$.

2. Write the equation of a locus which is symmetrical with respect to the x-axis, but not with respect to the y-axis.

3. Reduce the equation $4x - 7y = 14$ to the form $x \cos \alpha + y \sin \alpha = p$. What is the distance from the origin to this line?

4. Find the intersections of the loci whose equations are $25x^2 + 64y^2 = 1600$ and $x^2 + y^2 = 36$.

5. Find the equation of a line through the point $(0, -4)$ if it is parallel to the line $2x - 5y = 6$.

6. Decide whether the points $(3, 4)$, $(5, 8)$ and $(-2, -6)$ are collinear.

5

AFTER PAGE 35

1. Do the points $(5, 5\sqrt{3})$ and $(\sqrt{51}, 7)$ lie on a circle with its center at the origin? Do the points $(6, \sqrt{2})$ and $(5, \sqrt{21})$ lie on such a circle?

2. Reduce the equation $5x + 8y - 12 = 0$ to the form $y = mx + b$. What is the slope of this line? what is the y-intercept?

3. Reduce the equation $3x - 7y + 14 = 0$ to the form $\frac{x}{a} + \frac{y}{b} = 1$. What are the intercepts on the axes of this line?

4. Write the equation of the line through the point $(-2, 5)$ which is parallel to the line $2x - 5y + 7 = 0$.

5. What is the value of a if the line through the points $(a, 6)$ and $(4, -3)$ has a slope 2?

6. Find the distance from the origin to the line $-x + 5y - 9 = 0$.

6

AFTER PAGE 46

1. Find the projections on the axes of the segment connecting the points (x_1^2, y_1) and $[(x_1 + b)^2, y_1 + k]$.

2. Find the points in which the locus of $y^2 = 6x$ meets the locus of $x^2 + y^2 = 16$.

3. Write the equation of the line through the points $(2, -7)$ and $(-4, 6)$.

4. Write the equation of a line through the point $(-9, 4)$ which has the slope $-\frac{3}{7}$.

5. Reduce the equation $\frac{x}{4} + \frac{y}{7} = 1$ to the form $x \cos \alpha + y \sin \alpha = p$.

6. Find the distance from the point $P(2, 4)$ to the line $5x - 7y + 12 = 0$. Do the origin and the point P lie on the same or on opposite sides of this line?

7

AFTER PAGE 47

1. Discuss the lines represented by the equation $y = mx + b$ if b is fixed and m takes different values.

2. Verify that the line $4x - 3y + 7 = 0$ passes through the point $(2, 5)$; write this equation in the form $y - 5 = m(x - 2)$.

3. Find the value of C if the distance from the origin to the line $-3x + 8y + C = 0$ is 6.

4. The base of a triangle is on the line $3x - 9y + 16 = 0$ and $(8, 5)$ is the opposite vertex; find the altitude drawn from this vertex.

5. Under what conditions do the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ make a right angle?

6. Find the tangent of the acute angle formed by the lines $x + 4y - 9 = 0$ and $4x - 3y + 8 = 0$.

8

AFTER PAGE 47

1. Find the sum of the projections of the segments AB, BC, and CD on the y-axis if the points are: $A(h, k)$, $B(5, 2m)$, $C(4a, 7)$, $D(\sqrt{h}, 3k)$.

2. Find the point P if B is the middle point of the segment AP, the points A and B being $(-7, 1)$ and $(8, -4)$.

3. The segment AB is extended $\frac{3}{5}$ its length to P; find the coördinates of P if A and B are the points $(2, -9)$ and $(-8, 3)$.

4. Write the equation of a locus which is symmetrical with respect to the y-axis but not with respect to the x-axis.

5. The distance from the origin to the line $\frac{x}{6} + \frac{y}{b} = 1$ is 4; find the value of b if it is known to be positive.

6. Find the slopes of the bisectors of the angles formed by the lines $3x - 7y + 10 = 0$ and $-7x + 3y - 9 = 0$.

9

AFTER PAGE 50

1. Find the distance between the points in which $2x - 3y + 6 = 0$ cuts the circle $x^2 + y^2 = 25$.

2. The points $(3, 7)$, $(-2, 6)$, and $(4, -10)$ are the vertices of a triangle; find the altitude drawn from $(3, 7)$ to the opposite side.

3. Decide whether the lines $3x - 7y + 12 = 0$ and $2x + 5y - 8 = 0$ form a right angle.

4. Find the tangents of the angles formed by the lines $-x + 4y - 7 = 0$ and $3x + 2y + 6 = 0$.

5. Find the equation of the line through the intersection point of $x - 4y - 9 = 0$ and $2x + 5y + 12 = 0$ which passes through the point $(2, 7)$. Do not solve the equations simultaneously.

6. In the triangle in Ex. 2 find the equation of the line through the vertex $(-2, 6)$ which is parallel to the opposite side.

10

AFTER PAGE 68

1. Find the equation of the circle whose center is $(-7, 2)$ and whose radius is 6.

2. Find the center and the radius of the circle $x^2 + y^2 + 6x - 10y + 12 = 0$.

3. Find the equation of the circle circumscribed about the triangle whose vertices are (4, 2), (0, 6) and (2, 8).

4. Explain the meaning of the constants in the equations $y = mx + b$ and $\frac{x}{a} + \frac{y}{b} = 1$.

5. Find the equations of the bisectors of the angles formed by the lines $5x - 2y + 10 = 0$ and $x + 7y + 4 = 0$. Find the distance from the origin to each of these lines.

6. Find the equation of the line which passes through the origin and through the intersection of the lines $5x - 2y + 10 = 0$ and $x + 7y + 4 = 0$.

11

AFTER PAGE 68

1. Find the equation of the circle which passes through the points $(-7, 3)$, $(1, 1)$, and $(4, 0)$.

2. Find the distance from the point $(-2, 6)$ to the line whose slope is 2 and which passes through the point (2, 1).

3. Find the equation of the line through the intersection of $2x + 3y - 14 = 0$ and $3x - 2y + 8 = 0$ which is perpendicular to the line $6x - 5y + 12 = 0$.

4. Find the tangents of the angles formed by two lines whose slopes are $\frac{3}{4}$ and $-\frac{1}{3}$.

5. Find the equation of the circle whose center is $(5, -2)$ and which is tangent to the y-axis.

6. Find the center and the radius of the circle $2x^2 + 2y^2 - 4x + 8y - 16 = 0$.

12

AFTER PAGE 68

1. Write the equation of a locus which is symmetrical with respect to both axes. Is this locus symmetrical with respect to the origin?

2. Find the projections on the two axes of the segment joining the points (x_1^3, y_1^3) and $[(x_1 - h)^3, (y_1 + k)^3]$.

3. Find the centers and radii of the circles $x^2 + y^2 + 10x - 4y - 12 = 0$ and $4x^2 + 4y^2 + 8x - 16y + 10 = 0$. Also find the radical axis of these circles.

4. Find the equation of the line through the points (2, 7) and (9, b).

5. Find the equation of the bisectors of the angles formed by the lines $5x - 2y - 12 = 0$ and $-2x + 14y - 21 = 0$.

6. What is the value of A if the lines $2x - 5y + 10 = 0$ and $Ax + 3y - 12 = 0$ are parallel?

13

AFTER PAGE 68

1. Find the area of the triangle whose sides are $\frac{x}{7} + \frac{y}{10} = 1$, $x = 0$, and $y = 0$.

2. The line through the points (4, 7) and (− 2, 4) meets the curve $y = 4x^2 - 10$ in the points A and B; find the coördinates of these points.

3. Find the equation of the line through the intersection of $2x - 7y - 10 = 0$ and $4x + 5y + 12 = 0$ which passes through the point (− 1, 6); what is the distance from the origin to this line?

4. Find the center and the radius of the circle $Ax^2 + Ay^2 + Dx + Ey + C = 0$.

5. Find the equations of the tangents to the circle $x^2 + y^2 = 49$ which have the slope $-\frac{2}{3}$.

6. Find the equation of the tangent to the circle $x^2 + y^2 = 100$ at the point (− 8, 6).

14

AFTER PAGE 68

1. The point P divides the line connecting the points $A(2, -4)$ and $B(-7, 2)$ externally so that $AP : PB = -\frac{2}{5}$. Find the equation of a circle with radius 8 and P as its center.

2. What is the value of m if the line $y = mx + b$ makes with the line $y = \frac{3}{4}x - 10$ an angle whose tangent is 1?

3. Find the distance from the point (8, 10) to the line which is tangent to the circle $x^2 + y^2 = 25$ at the point (4, 3).

4. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ which pass through the point (6, 8); what is the length of one of these tangents from this point?

5. Find the equations of tangents with slope $-\frac{3}{7}$ to the circle $x^2 + y^2 = 40$.

6. Verify that the point (2, 4) lies on the circle $x^2 + y^2 - 10x + 4y = 16$. Find the equation of the tangent to this circle at this point.

15

AFTER PAGE 68

1. The point P divides the segment AB in the ratio $\frac{3}{7}$; find the equation of the line having a slope $\frac{3}{8}$ which passes through P, the points A and B being $(-2, 4)$ and $(8, -6)$.

2. Find the equation of the tangents to the circle $x^2 + y^2 = 49$ which are perpendicular to the line $-3x + 7y - 10 = 0$.

3. Verify that the point $(1, 3)$ is on the circle $x^2 + y^2 + 10x - 12y + 16 = 0$. Find the equation of the tangent to this circle at this point, and find the equations of the bisectors of the angles formed by this tangent and the line joining the point $(1, 3)$ to the center of the circle.

4. Find the equations of the tangents to the circle $x^2 + y^2 = 36$ at the points for which $x = 3$.

5. Find the length of the tangent from the point $(-7, 8)$ to the circle $2x^2 + 2y^2 - 8x + 12y - 20 = 0$.

6. Find the equations of the tangents to the circle $x^2 + y^2 - 8x + 4y - 5 = 0$ which pass through the point $(3, 5)$.

16

AFTER PAGE 75

1. The line $7x - 3y = 12$ meets the circle $x^2 + y^2 = 36$ in the points P and Q; find the projections of the segment PQ on the two axes.

2. Find the equation of the line through the intersection of $x - 3y + 10 = 0$ and $4x + y - 12 = 0$ which is parallel to the line $7x - y + 14 = 0$.

3. Find the radical axis of the circles $x^2 + y^2 - 3x + 5y - 12 = 0$ and $x^2 + y^2 + 6x - 3y + 8 = 0$; find the distance from the origin to this line.

4. Find the equation of the circle through the intersection points of the circles in Ex. 3 which passes through the point $(3, 7)$.

5. Find the intersection points of the two parabolas $y^2 = 8x$ and $x^2 = 8y$.

6. Find the length of the latus rectum of the parabola $y^2 = 12x$.

17

AFTER PAGE 75

1. Write the equation of the line through the points $(-1, 7)$ and $(8, -2)$; find two other points on this line and verify that the slope of the line connecting them is the slope of this line.

2. Find the distance from the point (x_1, y_1) to the line $\frac{x}{a} + \frac{y}{b} = 1$.

3. Find the equation of the tangent to the circle $x^2 + y^2 = 49$ at the point $x = 5$, $y = 2\sqrt{6}$. Find the slope of this tangent and substitute in $y = mx + 7\sqrt{1 + m^2}$ to verify that the two methods give the same equation.

4. Develop the equation of the parabola of which $x = -4$ is the directrix and $(4, 0)$ is the focus.

5. Find the equations of the tangents to the circle $x^2 + y^2 - 10x + 7y = 19$ at the points for which $x = 5$.

6. Find the equations of the tangents to the circle $x^2 + y^2 + 4x + 8y - 5 = 0$ which pass through the point $(8, 1)$; find the length of the tangents from this point to the circle.

18

AFTER PAGE 81

1. Discuss the locus of $x^2 + y^4 = 10$ as to symmetry.

2. Find the equation of the line through $(-1, 6)$ which is perpendicular to the line $x + 7y - 12 = 0$.

3. Find the tangents of the angles which the bisector of the acute angles formed by $7x - y + 4 = 0$ and $3x + 5y - 12 = 0$ forms with the line $3x + 5y - 12 = 0$.

4. Find the equation of the circle through the point $(3, 5)$ which passes through the intersection points of the circles $x^2 + y^2 + 7x - 5y - 5 = 0$ and $x^2 + y^2 - 7x + 5y - 8 = 0$; find the center and radius of this circle.

5. Find the equation of the form $y^2 = 4px$ of a parabola which passes through the point $(2, 4)$; find the equation of the tangent to this parabola which has a slope $\frac{2}{7}$.

6. Find the equations of the tangents and the normals to the parabola $y^2 = 12x$ at the points for which $x = 6$.

19

AFTER PAGE 81

1. Find the equations of the tangents to the circle $x^2 + y^2 = 81$ whose slopes are $\frac{2}{3}$; reduce the equations to the form $x \cos \alpha + y \sin \alpha = p$ and verify that the distance from the origin to each line is 9.

2. Find the value of p if the parabola $x^2 = 4py$ passes through the point $(6, 6)$; find the focus and the length of the latus rectum of this parabola.

3. Find the equations of the lines through the intersection point of $x + y + 1 = 0$ and $3x - 7y + 8 = 0$ which are tangent to the parabola $y^2 = 8x$.

SUGGESTION: Note that $p = 2$. Write $x + y + 1 + k(3x - 7y + 8) = 0$ and reduce to the form $y = mx + \frac{2}{m}$.

4. Find the equations of the tangents and normals to the parabola $y^2 = 12x$ at the points for which $x = 3$.

5. Derive the equation $x_1x + y_1y = r^2$ of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) on it.

6. Derive the equation $x_1x + y_1y + D(x + x_1) + E(y + y_1) + C = 0$ of the tangent to the circle $x^2 + y^2 + 2Dx + 2Ey + C = 0$ at the point (x_1, y_1) on it.

20

AFTER PAGE 81

1. Tangents are drawn from the point (6, 1) to the circle $x^2 + y^2 - 6x - 4y + 8 = 0$; find the points of contact and the projections on the axes of the chord joining these points.

2. Find the equation of the circle with center $(-5, 6)$ to which the line $-3x + 2y = 8$ is a tangent.

3. Find the equation of the circle which passes through the points (2, 5), $(-4, 2)$, and $(6, -3)$; also find the equation of the tangent to this circle at each of these points.

4. Find the equation of a circle which passes through the intersection points of the circles $x^2 + y^2 - 8x - 2y + 12 = 0$ and $x^2 + y^2 + 8x + 2y - 20 = 0$ if its center is on the line $y = x - 9$.

5. Find the value of p if the line $2x - 9y - 12 = 0$ is tangent to the parabola $y^2 = 4px$.

SUGGESTION: Reduce the equation to the form $y = mx + \frac{p}{m}$.

6. What is the directrix of the parabola $y^2 = 10x$? State the relation between the normal, the focal radius, and a line parallel to the axis, all drawn through the same point on a parabola.

21

AFTER PAGE 81

1. Find the equation of the normal and of the focal radius of the parabola $y^2 = 8x$ at the point (2, 4) and find the tangent of the angle formed

by these lines; also find the tangent of the angle formed by the normal and the line $y = 4$ and compare.

2. Find the directrix and focus of the parabola $y^2 = 4px$ which passes through the point $(-8, 8)$.

3. Write the equation of the line with slope $-\frac{2}{7}$ which passes through the point $(-2, 6)$; reduce the equation to the form $y = mx + b$. What is the y-intercept of this line?

4. Find the values of D if the circle $x^2 + y^2 + 2Dx - 6y + 22 = 0$ has a radius 6; find the length of a tangent from the point $(8, -5)$ to this circle.

5. Find the values of D and E if the line $6x + 4y + 11 = 0$ is the radical axis of the circles $x^2 + y^2 + 9Dx + 2Ey + 12 = 0$ and $x^2 + y^2 - 4x + 6y - 10 = 0$.

6. The focus of a parabola is $(-2, 0)$ and its directrix is $x = 2$; develop the equation using the definition of the parabola; find the tangents to this parabola at the points for which $x = -1$.

22

AFTER PAGE 81

1. A line is parallel to the line $-4x + 3y - 12 = 0$ and bisects the segment connecting $(-1, 6)$ and $(4, -2)$; find its equation and find the distance from the point $(-8, 3)$ to this line.

2. A line with slope $\frac{3}{2}$ is tangent to the circle $x^2 + y^2 = 25$ at a point for which x is positive; find the equations of the lines which bisect the angles formed by this tangent and the radius drawn to the point of tangency.

3. Find the equations of the tangents to the circle $x^2 + y^2 + x - 12y = 0$ at the points for which $x = 4$.

4. The parabola $y^2 = 4px$ passes through the point $(5, 10)$; find the slopes of the tangent and the normal to this parabola at the point for which $y = 6$. Write the equations of these lines.

5. Write the equation of the circle of which the latus rectum of the parabola $y^2 = 12x$ is the diameter; find the equations of the tangents from the point $(-4, 6)$ to the circle.

6. Prove that $y = mx + \frac{p}{m}$ is tangent to the parabola $y^2 = 4px$ for all values of m.

23

AFTER PAGE 81

1. The line $x - y = 6$ cuts the parabola $y^2 = x$ in the points P and Q; normals to the parabola at P and Q meet in R; find the equation of the line through R which is perpendicular to PQ.

2. Find the distance from the origin to the tangent to the parabola $y^2 = 6x$ if the slope of the tangent is $-\frac{1}{5}$.

3. Find the equations of the tangents to the circle $x^2 + y^2 = 100$ at the points for which $y = 8$.

4. Prove that the circle whose diameter is the latus rectum of a parabola is tangent to the directrix.

5. Construct in order the following lines related to the parabola $y^2 = 16x$ on which (x_1, y_1) is a point: (a) the focal radius to this point, (b) the line $y = y_1$, (c) the normal to the parabola, (d) the tangent to the parabola.

6. Find the length of the altitude drawn from the vertex $(4, -1)$ in the triangle whose vertices are $(4, -1)$, $(-2, 3)$, and $(6, -4)$.

24

AFTER PAGE 92

1. Find the equation of a line through the point $(2, -9)$ having a slope $-\frac{7}{9}$. What is the tangent of the angle formed by this line and the line $2x - 3y + 4 = 0$.

2. In the circle $x^2 + y^2 + 9x - 7y = 0$ find the projections on the axes of the segment joining the center with the point $(6, -2)$; what is the sum of the squares of these projections? what is the radius of the circle?

3. Find the equation of the circle which passes through the intersection points of $x^2 + y^2 - 2x + 2y - 10 = 0$ and $x^2 + y^2 + 6x - 4y - 12 = 0$ and also through the point $(6, 8)$; find the tangent to the circle through this last point.

4. In the ellipse $\frac{x^2}{49} + \frac{y^2}{16} = 1$, find, (a) the foci, (b) the directrices, (c) the length of latus rectum, (d) the sum of the focal radii drawn to a point on the ellipse.

5. In developing the equation of the ellipse how is the value of b determined?

6. Find the equations of lines with slope $\frac{2\sqrt{3}}{9}$ which are tangent to the ellipse $16x^2 + 36y^2 = 576$; find the point of contact of one of these tangents and then write its equation in the form $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$.

25

AFTER PAGE 92

1. Find the altitude upon the side BC of the triangle whose vertices A, B, C are $(2, 9)$, $(-4, 3)$, $(5, -4)$.

2. Find the equation of the circle circumscribed about the triangle in Ex. 1.

3. The parabola $y^2 = 4px$ passes through the point $(-2, 8)$; find the equation of the tangent to it which has a slope $\frac{3}{7}$.

4. Find the foci and directrices of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ $(a > b)$ whose eccentricity is $\frac{1}{2}\sqrt{3}$ and whose semiminor axis is 7.

5. Find the equations of lines having a slope $16/15$ which are tangent to the ellipse $16x^2 + 25y^2 = 400$; find the point of contact of one of these and verify that the tangent at the point (x_1, y_1) has the slope $-\frac{b^2}{a^2}\cdot\frac{x_1}{y_1}$.

6. Find the equations of the tangents to the parabola $y^2 = 20x$ at the points for which $x = 5$.

26

AFTER PAGE 92

1. Find the equation of the line which is perpendicular to the line $4(x + 5) - 3(5 - 2y) + 6 = 0$ and whose y-intercept is -3.

2. Write the equation of the circle whose center is $(2\sqrt{3}, 4)$ and which passes through the point $(5\sqrt{3}, -3)$.

3. The radical axis of the two circles (1) $x^2 + y^2 = 19$, (2) $x^2 + y^2 - 14x + 2Ey + 28 = 0$ passes through the point $(5, -23/4)$; find the equation of the tangent to (1) at a point in which the radical axis meets it, also find the equation of the tangent from $(5, -23/4)$ to (2).

4. Find the slopes of the tangent and the normal to the parabola $y^2 = -16x$ at the point for which $y = 8$.

5. Develop the equation of the ellipse whose major axis is 10 and whose eccentricity is $\frac{3}{4}$.

6. Find the latus rectum of an ellipse whose minor axis is 6 and whose eccentricity is $\frac{3}{5}$.

27

AFTER PAGE 92

1. Note that $x\cos\alpha + y\sin\alpha = 5$ is tangent to the circle $x^2 + y^2 = 25$ for all values of α. Reduce this equation to the form $y = mx + 5\sqrt{1+m^2}$.

SUGGESTION: How is α related to the angle which the line makes with the x-axis?

2. Find the coördinates of the point P which divides the segment connecting the points $(\sqrt{a}, 2b)$ and $(2a, 3\sqrt{b})$ in the ratio $-\frac{8}{3}$.

3. Find the equations of the bisectors of the angles formed by the tangent and normal to the ellipse $\frac{x^2}{49} + \frac{y^2}{25} = 1$ at the point $(5\frac{3}{5}, 3)$.

4. Find the directrix and length of latus rectum of the parabola $y^2 = 36x$; also find the equations of the tangents at the points for which $x = 4$.

5. In an ellipse the foci are $(\pm 3, 0)$ and the directrices $x = \pm 5\frac{1}{3}$; find the sum of the focal radii drawn to a point on it.

6. Find the length of the tangent from $(-10, 2)$ to the circle $x^2 + y^2 - 14x - 2y + 20 = 0$.

28

AFTER PAGE 96

1. Find the equation of the ellipse whose foci are $(\pm 4, 0)$ and whose semiminor axis is 3; find the equations of tangents to this ellipse whose slopes are $-\frac{3}{5}$.

2. At a point (x_1, y_1) on an ellipse construct in order (a) the focal radii, (b) the normal, (c) the tangent.

3. Find the equation of the parabola whose focus is $(6, 0)$ and whose directrix is $x = -6$; find the equation of the tangent to the parabola whose slope is $\frac{3}{10}$.

4. The point $(p, 0)$ is the focus of a parabola and (x_1, y_1) is a point on it; construct in order (a) the focal radius to this point, (b) the line $y = y_1$, (c) the normal to the parabola, (d) the tangent to it.

5. Find the equations of the tangents to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the points for which $x = 2$.

6. Find the equation of the circle which passes through the point (10, 4) and through both intersections of the circles $x^2 + y^2 + 2x + 4y - 40 = 0$ and $x^2 + y^2 - 8x - 12y - 30 = 0$; also find the tangent of the angles formed by the line $x - 4y + 2 = 0$ and the radical axis of these circles.

29

AFTER PAGE 96

1. The lines $x - y + 10 = 0$, $3x + y - 2 = 0$, and $x - 8y - 10 = 0$ form a triangle; find its altitudes.

2. The sum of the focal radii of an ellipse is 10 and the directrices are $x = \pm 6$; find the equation of the ellipse.

3. Find the equations of the lines through the vertices of the triangle in Ex. 1 which are parallel to the opposite sides.

4. Find the equation of the form $x^2 = 4py$ of the parabola which passes through the point (12, 6).

5. The foci of the ellipse $\frac{x^2}{49} + \frac{y^2}{b^2} = 1$ are $(\pm \sqrt{33}, 0)$; find the value of b.

6. The latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{25} = 1$ is $8\frac{1}{5}$; find the value of a.

30

AFTER PAGE 110

1. In the hyperbola $\frac{x^2}{64} - \frac{y^2}{36} = 1$ find the eccentricity, directrices, foci, and the latus rectum. Compare the eccentricities of the ellipse and the hyperbola.

2. Find the center and radius of the circle $x^2 + y^2 - 4x + 10y - 20 = 0$; also find the equations of the tangents to this circle from the point (5, 2).

3. Find the projections on lines parallel to the axes of the segment connecting the points $(\sqrt{x_1}, y_1)$ and $(\sqrt{x_1 + b}, y_1 + k)$.

4. Find the equation of the tangent to $y^2 = -10x$ at the point for which $y = 10$; find the equations of the tangents with slopes $\frac{2}{7}$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

5. Find the equations of the tangents to $x^2 + y^2 = 16$ at the points for which $y = 3$.

6. Given the foci of an ellipse and a point on it construct in order (a) the focal radii, (b) the normal, (c) the tangent at the given point.

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1. In the hyperbola $\frac{x^2}{16} - \frac{y^2}{4} = 1$ find the eccentricity, foci, directrices, latus rectum, and the difference of the focal radii. Compare the last of these with the relation between focal radii in the ellipse.

2. Find the equation of the line through the origin which is perpendicular to the line whose intercepts on the axes are -4, 7.

3. In the ellipse $\frac{x^2}{100} + \frac{y^2}{16} = 1$ find the directrices; also find the tangents at the points for which $x = 8$. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ what is the slope of the normal at the point (x_1, y_1)?

4. Give the definition of the radical axis of two circles. What is the equation of the tangent to the circle $x^2 + y^2 + 2Dx + 2Ey + C = 0$ at a point (x_1, y_1) on it.

5. Derive the formula giving the slope of the normal to the parabola $y^2 = 4px$ at the point for which $y = y_1$.

6. In developing the equation of the hyperbola how is b determined?

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AFTER PAGE 110

1. In the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ the difference between the focal radii is 8, and the latus rectum is 18; find the equations of the tangents whose slopes are $\frac{5}{3}$. Compare the formula for tangent with a given slope with the corresponding formula for the ellipse.

2. Compare formulas for foci, directrices, latus rectum, and eccentricity for the ellipse and the hyperbola.

3. Derive the equation of the directrix and the formula giving the length of the latus rectum of the parabola $x^2 = 8y$.

4. Find the equation of the circle circumscribed about the triangle whose

vertices are (0, 0), (0, 4), and (3, 6); find the equations of the tangents to this circle whose slopes are $\frac{2}{7}$.

5. Derive the equation of the line with slope m which passes through the point (x_1, y_1).

6. Prove that a tangent to an ellipse bisects one pair of the angles formed by the focal radii drawn to the point of tangency.

33

AFTER PAGE 110

1. In the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the point $(5\sqrt{5}, 0)$ is a focus and $x = \frac{9}{5}\sqrt{5}$ is a directrix; find the eccentricity, the length of the latus rectum, and the difference of two focal radii. Also find the equations of the tangents to this hyperbola whose slopes are 3.

2. Find the equations of the normals to the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ at the points for which $y = 1$.

3. Prove that in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the sum of the focal radii drawn to a point on it is $2a$.

4. Find the equation of the tangent to the parabola $y^2 = -10x$ whose slope is $\frac{3}{10}$; find the distance from the origin to this tangent.

5. Prove the formula giving the distance between two points.

6. Find the lengths of the tangents to the circles $x^2 + y^2 - 10x - 10y - 20 = 0$ and $x^2 + y^2 + 2x + 4y - 40 = 0$ from a point on their radical axis.

34

AFTER PAGE 110

1. Find in two ways the equation of the line with slope $\frac{2}{5}$ which passes through the intersection point of $x - 3y + 12 = 0$ and $5x + 2y - 14 = 0$; compare the method of finding the equation of a circle through the intersection of two given circles.

2. Find the equations of the tangents whose slopes are $\frac{5}{3}$ to an ellipse if its foci are $(\pm \sqrt{39}, 0)$ and its directrices are $x = \pm \frac{64}{\sqrt{39}}$.

3. State a relation between the angles formed by the focal radius and the axis of a parabola, and the angle formed by the focal radius and the normal. State a relation between the angles made by the focal radii and the normal to an ellipse; make a similar statement for the hyperbola.

4. Give the eccentricity of the ellipse and of the hyperbola in terms of the semiaxes.

5. In the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the difference of the focal radii is 6 and the length of the latus rectum is $10\frac{2}{3}$; find the eccentricity and the directrices.

6. Find the equations of the tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{4} = 1$ whose slopes are 3; also find the equations of the tangents at the points for which $x = -6$.

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AFTER PAGE 115

1. Find the distance from the point (2, 7) to a line through (− 4, 3) which is parallel to $x - 9y + 10 = 0$.

2. Give the equation of the tangent through the point (x_1, y_1) to each of the curves (a) $x^2 + y^2 = r^2$, (b) $y^2 = 4px$, (c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (d) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the point (x_1, y_1) being on each curve.

3. In the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$ find the directrices, foci, eccentricity, latus rectum, and asymptotes.

4. Define a rectangular hyperbola. What is the eccentricity of the ellipse $\frac{x^2}{40} + \frac{y^2}{20} = 1$? What are the slopes of the normals to it at the points for which $y = 4$?

5. Find the equation of the circle which passes through the points (−4, 6), (8, − 2), and (0, 7).

6. Derive the formula giving the length of the latus rectum of the ellipse.

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AFTER PAGE 115

1. State and derive the equation of the bisectors of the angles formed by two intersecting lines.

2. The line $y = \frac{3}{5}x$ is an asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $x = \frac{25}{34}\sqrt{34}$ is a directrix; find the foci, the difference of the focal radii, and the equations of the tangents whose slopes are $-\frac{5}{2}$.

3. The sum of the focal radii of an ellipse is 16 and the semiminor axis is 5; find the equation of the ellipse.

4. The equilateral hyperbola $x^2 - y^2 = 36$ passes through the point (10, 8). Show that the tangent at this point makes equal angles with the focal radii.

5. Find the equations of the tangents to the circle $x^2 + y^2 = 81$ which are parallel to the line $3x - y + 12 = 0$.

6. Derive the formula giving the length of the latus rectum of the parabola $y^2 = 4px$.

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AFTER PAGE 115

1. Find the equation of the line which passes through the intersection point of $3x - y + 10 = 0$ and $x + 6y - 10 = 0$ and through the point P which divides the segment connecting (0, 4) and (8, 0) externally in the ratio $-\frac{2}{3}$.

2. Find the equations of (a) the tangents to $x^2 + y^2 - 6x + 4y - 10 = 0$ at the points for which $x = 2$, (b) the tangent to the parabola $y^2 = 16x$ having a slope -3, (c) the tangents to $\frac{x^2}{36} - \frac{y^2}{9} = 1$ at the points for which $x = 8$.

3. It is given that $y = mx + \sqrt{4m^2 + b^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m; find the equations of the directrices if it is also given that the eccentricity is $\frac{5}{7}$.

4. Given the foci of an ellipse and one point on it; construct the normal and the tangent to the ellipse at this point.

5. The line $y = \frac{3}{4}x$ is an asymptote of a hyperbola and $x = \frac{32}{5}$ is a directrix; find the foci and the length of the latus rectum.

6. Find the intersection points of the rectangular hyperbola whose foci are $(\pm 4\sqrt{2}, 0)$ and the parabola $y^2 = 6x$.

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AFTER PAGE 115

1. Write the equation of the circle whose center is $(2, -9)$ and whose radius is 8; find the equation of the line through the center of this circle which when reduced to the form $x\cos\alpha + y\sin\alpha = p$ gives $\alpha = -45°$.

2. The semiconjugate axis of a hyperbola is 4, the difference between two focal radii is 16; find the equation of the hyperbola and also its eccentricity.

3. Give the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) on it; what is the slope of this tangent?

4. Give in terms of a and b the eccentricity and the latus rectum of an ellipse.

5. Derive the equations of the asymptotes of a hyperbola.

6. Given the foci of a hyperbola and one point on it construct in order (a) the focal radii, (b) the tangent, (c) the normal at this point.

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AFTER PAGE 115

1. The length of the latus rectum of a hyperbola is $\frac{32}{7}$ and $y = -\frac{4}{7}x$ is an asymptote; find the foci, directrices, and the equations of tangents with slope $\frac{3}{4}$.

2. Find the equation of the circle through the intersection points of $x^2 + y^2 - 2x + 10y - 40 = 0$ and $x^2 + y^2 + 8x - 2y - 20 = 0$ which passes through the point $(12, 0)$; also find the equations of the tangents to this circle which pass through the point $(12, 0)$.

3. Find the equation of an ellipse if the sum of the focal radii is 14 and its eccentricity is $\frac{1}{7}\sqrt{33}$.

4. Define a rectangular hyperbola; at what angles do the asymptotes intersect?

5. Derive the formulas giving in terms of p and the ordinates of the point of tangency, the slopes of the tangent and the normal of the parabola $y^2 = 4px$.

6. Give the definition of the eccentricity of an ellipse and express it in terms of the semiaxes of the ellipse.

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AFTER PAGE 115

1. Define the hyperbola; give the eccentricity in terms of the semiaxes; give the equation of the tangent at a point (x_1, y_1) on it; state the relations between the focal radii, tangent, and normal at a point on it.

2. Define the ellipse; give the equations of the directrices, in terms of the semiaxes; give the equations of the tangents with slope m; state the relation between the focal radii, the tangent, and the normal at a point on it.

3. Give general directions for finding the intersections of curves whose equations are given. How may a tangent to a curve through a point not on it be found?

4. Find the center and the radius of the circle $x^2 + y^2 - 7x + 9y - 48 = 0$.

5. Derive the equation of the tangent to $x^2 + y^2 = r^2$ at a point (x_1, y_1) on the circle.

6. Derive the equation of the tangent to the parabola $x^2 = 4py$ at the point (x_1, y_1) on it.

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AFTER PAGE 115

1. Find the equation of a tangent with slope -3 to the circle which passes through the points $(0, 6)$, $(-4, 2)$, and $(-2, 0)$; find the point (x_1, y_1) of contact of this line and reduce its equation to the form $y - y_1 = m(x - x_1)$.

2. Find the equation of an ellipse whose eccentricity is $\frac{1}{2}\sqrt{2}$ and whose foci are $(\pm 7, 0)$; find the length of the latus rectum of this ellipse, and also find the equations of the tangents to it at the points for which $x = 3$.

3. The asymptotes of a hyperbola are $y = \pm \frac{7}{6}x$ and its directrices are $x = \pm 4$; find its foci, and the difference between its focal radii.

4. From a point P on a rectangular hyperbola draw perpendiculars to its asymptotes and prove that the product of these is a constant.

5. Find the equation of the tangent to $xy = k$ at a point (x_1, y_1) on the curve. Find the area of the triangle formed by this tangent and the coördinate axes.

6. State the relation between a pair of conjugate hyperbolas; what is the relation between their asymptotes?

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AFTER PAGE 115

1. Find the distance from the point (3, 4) to the radical axis of the circles $x^2 + y^2 - 14x + 4y - 16 = 0$ and $x^2 + y^2 + 14x - 4y - 16 = 0$.

2. Give the length of the latus rectum (a) in the parabola $y^2 = -10x$, (b) in the hyperbola $\frac{x^2}{49} - \frac{y^2}{4} = 1$.

3. The semimajor axis of an ellipse is 5 and its eccentricity is $\frac{3}{7}$; find the equations of the tangents to it whose slopes are $\frac{2}{3}$; also find the slope of the tangent at the point (x_1, y_1) on the ellipse.

4. The line $y = \frac{3}{4}x$ is an asymptote of a hyperbola and $y = 2x + \sqrt{73}$ is a tangent to it. Find the equation of the hyperbola and its eccentricity.

5. Find the equation of the rectangular hyperbola which passes through the point (8, 6), using the axes of the hyperbola as coördinate axes.

6. Find the equation of the rectangular hyperbola which passes through the point (8, 3), using its asymptotes as axes.

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AFTER PAGE 115

1. Prove that $A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0$ is the equation of a line which passes through the intersection point of $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$.

2. State the formula giving the eccentricity of an ellipse in terms of its semiaxes; prove this formula.

3. State the formulas giving the foci and directrices of a hyperbola, and the tangent at a point on it.

4. Discuss a rectangular hyperbola and its conjugate; what are the asymptotes of these hyperbolas?

5. Find the equations of the tangents to the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ at the points for which $y = 3$.

6. Prove that in an ellipse the normal bisects the angle formed by the focal radii; state the corresponding proposition for the parabola.

44

AFTER PAGE 115

1. Prove that the line passing through the centers of any two circles of the set of circles which passes through the intersection points of $x^2 + y^2 + 2D_1x + 2E_1y + C_1 = 0$ and $x^2 + y^2 + 2D_2x + 2E_2y + C_2 = 0$ is perpendicular to the radical axis of these circles.

2. Write the equations of tangents with slope m, (a) to the circle $x^2 + y^2 = r^2$, (b) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (c) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

3. The sum of the focal radii of an ellipse is 18 and the foci are $(\pm 7, 0)$; find the tangents to this ellipse at the points for which $x = 2$.

4. The foci of a hyperbola are $(\pm 10, 0)$ and $x = \frac{18}{5}$ is a directrix; find the equation of the hyperbola and the equations of its asymptotes. Find the equation of the hyperbola which is conjugate to this one.

5. In the parabola $y^2 = 4px$ what is the equation of the directrix? What are the coördinates of the focus?

6. Give a set of points related to a hyperbola which makes it possible to construct a tangent and a normal to it at a given point on it.

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AFTER PAGE 115

1. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = 4$ and $e = \frac{1}{3}$; find b; also find the foci, directrices, and latus rectum.

2. Find the equations of tangents with slope $\frac{7}{3}$, (a) to the circle $x^2 + y^2 = r^2$, (b) to the parabola $y^2 = 4px$, (c) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (d) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

3. Write the equation of the tangent to each of the following curves at the point (x_1, y_1) on it: (a) $y^2 = 4px$, (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

4. The line $3x - 2y + 6 = 0$ is the bisector of angles formed by the lines $x - 3y + 12 = 0$ and $Ax - 3y + C = 0$; find the values of A and C.

5. Find the equations of the tangents to the equilateral hyperbola $xy = 8$ whose slopes are -2.

6. A hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point $(7\frac{1}{2}, 3)$ and its conjugate hyperbola passes through the point, $(8, 6\frac{2}{3})$; find the values of a and b.

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AFTER PAGE 115

1. Derive the equation of the parabola whose focus is $(0, p)$ and whose directrix is $y = -p$; also derive the equation of the tangent to it whose slope is m.

2. Derive the equation of the hyperbola whose vertices are $(\pm a, 0)$ and whose eccentricity is e. Define b and give e in terms of a and b; give the coördinates of the foci, the difference of two focal radii, and the equations of tangents with slope m.

3. Give the relations between focal radii, normal, and tangent (a) in case of the ellipse, (b) in case of the hyperbola.

4. Find the equations of the tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the points for which $y = -3$.

5. Find the equation of the normal to the curve $xy = 20$ at the point for which $x = 5$.

6. What is the equation of the hyperbola which is conjugate to $xy = 10$?

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AFTER PAGE 115

1. The eccentricity of a hyperbola is $\frac{13}{12}$ and its foci are $(\pm 13, 0)$; find the difference of its focal radii and the equations of its asymptotes.

2. The sum of the focal radii of an ellipse is 26 and its eccentricity is $\frac{12}{13}$; find the equations of the tangents to it whose slopes are $-\frac{3}{4}$.

3. The line $x\cos 30° + y\sin 30° = p$ is tangent to the hyperbola $xy = 8$; find the value of p.

4. Find the equation of the tangent to the rectangular hyperbola whose major axis is 8 at the point (x_1, y_1) on it, the axes of the hyperbola being used as coördinate axes.

5. Find the equation of the hyperbola which is conjugate to $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

6. Find the length of the latus rectum of the hyperbola found in Ex. 5.

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AFTER PAGE 115

1. The eccentricity of a hyperbola is $\frac{5}{4}$ and its directrices are $x = \pm\frac{16}{5}$; find its equation.

2. The minor axis of an ellipse is 8 and its eccentricity is $\frac{5}{6}$; find the length of the major axis.

3. Show that as m approaches $\frac{b}{a}$ the equation $y = mx + \sqrt{a^2m^2 - b^2}$ approaches $y = \frac{b}{a}x$.

4. If the tangent to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is given and one of its foci, show how to find the other focus.

5. Find the equation of the normal to the hyperbola $x^2 - y^2 = k^2$ at the point (x_1, y_1) on it.

6. Find the equation of the tangent at the point (x_1, y_1) to the hyperbola which is conjugate to $x^2 - y^2 = k^2$, where (x_1, y_1) is a point on the hyperbola.

CHAPTER XVII

REVIEW OF SUBJECT MATTER OF PLANE ANALYTIC GEOMETRY

A CLEAR understanding of any well organized system of thought, such as analytic geometry, usually reveals an essential simplicity, which to a more casual observer is not apparent. The different ideas involved are found to be comparatively few in number, rather easily comprehended, and naturally and even obviously related. The purpose of this chapter is to take a general view of the subject we have studied in order to point out the dominant ideas and processes that are involved and the ways in which these are brought together to form one of the most effective and simple of the methods that have been devised for mathematical investigation. The more comprehensive understanding to which such a review should lead will serve not only to reveal the simplicity of the subject itself and its relations to what is studied earlier, but also to make remembering much more easy and certain.

200. The fundamental ideas and processes of analytic geometry.—The essential characteristic of analytic geometry is the use of algebraic equations to represent geometric loci. To make this possible it is necessary to have a system of coördinates by means of which points are represented by numbers. Historically, what amounts to a system of coördinates was used long before analytic geometry proper was invented (see page 292 in the Historical Sketch), and its practical equivalent is found in such devices as are used in describing the location of a point on the earth's surface by giving its latitude and longitude (see page 5) and in making graphical representation

of data such as is now in very general use not only in scientific writings but in popular magazines and even in advertisements.

The algebra used in an elementary course such as the present, rarely goes beyond that given in elementary courses in our high schools, and never beyond that given in freshman courses in "College Algebra."

Some of the simpler theorems from plane geometry are used and also a few formulas from trigonometry. These are enumerated in Chapter XIX of this book.

All the work in deriving equations and formulas, and in solving problems is essentially algebraic.

201. Points and numbers.—Logically analytic geometry has its beginning in representing each point on a straight line by means of a number. Since the line is endless in both directions this necessitates the use of signed numbers which may be regarded as starting with the number zero and extending indefinitely in two opposite "directions." The representation of a point in a plane by means of two numbers, its coördinates, is then obvious, as is also the representation of a point in space by three numbers (see chapter XVIII).

The next step is to notice that the length of any segment AB (on the x-axis) is equal to the difference between the abscissas of its end-points, having of course, proper regard to the signs of the abscissas. A clear understanding of this fact (see pages 2 and 3) removes not a few of the later difficulties. It follows immediately for instance, that the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ [1], giving the distance between *any two points*, (x_1, y_1), (x_2, y_2) is simply a translation into algebraic language, the language of analytic geometry, of the Pythagorean proposition, which states that the square of the hypotenuse of a right triangle is equal to the sum of the squares of its sides.

Using the most elementary theorems on equal triangles the formulas $x = \dfrac{x_1 + x_2}{2}$, $y = \dfrac{y_1 + y_2}{2}$ [2], giving the coördinates

of the middle point of a segment follow at once; and using the simplest properties of similar triangles we have $x = \frac{mx_2 + nx_1}{m + n}$, $y = \frac{my_2 + ny_1}{m + n}$ [3], giving the coördinates of a point dividing a segment in a given ratio.

202. Variables; the equation and its locus.—While a variable is an algebraic, or analytic, symbol its meaning is perhaps most readily apprehended in connection with a moving point. Suppose that the point $P(x,y)$ moves along the line $x + y = 6$. Then its coördinates x and y change or "vary," as the point moves. When the point moves from the point (0, 6) indefinitely toward the right (and of course also downward), the abscissa, x, takes any real positive value whatever, and as the point moves in the opposite direction the abscissa takes any negative value. That is, the variable x is free to take any real value whatever. This is also true of the variable y.

However, as the point $P(x,y)$ moves along this line the variables x and y vary subject to the condition that their algebraic sum is 6. It is for this reason that $x + y = 6$ is regarded as the equation of the line.

In general, if a point moves so that its coördinates satisfy a certain equation, then the point traces a curve which is the locus of the equation, and the equation is called the equation of the curve (see the statements in italics on page 17). Note that the word "curve" is used so as to include straight lines.

Since the coördinates of a point are real numbers the variables are, for the purpose of analytic geometry, restricted to real values. Thus, the equation $x^2 + y^2 = -1$ has no locus since it is satisfied by no pair of real numbers. We sometimes express this by saying that the equation has no "real locus."

If two equations $f_1(x, y) = 0$ and $f_2(x, y) = 0$ are solved simultaneously obtaining a solution x_1, y_1, then the point (x_1, y_1) lies on the loci of both equations. That is, it is an intersection

point of the loci. Hence solving simultaneously two equations in two unknowns (variables) amounts to the finding of the intersection points of the loci of the equations.

If two such equations have no real simultaneous solution, their loci do not meet.

203. The equation of a straight line.—The proof that the general first degree equation $Ax + By + C = 0$ is the equation of a straight line depends upon the fact that if $x_1, y_1; x_2, y_2; x_3, y_3$ satisfy this equation, then

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_3}{x_2 - x_3} \qquad [5]$$

Note that [5] is derived purely algebraically from the given data. That under this condition the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ lie in a straight line is a consequence of the properties of similar triangles and of parallel lines. See §§27, 179, 183, 184

The forms $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$, or $y - y_1 = (x - x_1)\frac{y_1 - y_2}{x_1 - x_2}$ [6], $y - y_1 = m(x - x_1)$ [7], $y = mx + b$ [8], and $\frac{x}{a} + \frac{y}{b} = 1$ [9], of the equation of a straight line are derived very easily from [5].

204. The normal form of the equation of a straight line.—The form $x \cos \alpha + y \sin \alpha = p$ [10] of the equation of the straight line is peculiar in that it involves the length of the perpendicular from the origin to the line and the angle α which this perpendicular makes with the x-axis. In this book the normal p from the origin upon a line is regarded as positive for all lines. In this respect compare the "distance" from the origin to a point used in the definition of trigonometric functions (see §250 page 281). As, for different lines, the normal swings in the counter-clockwise direction, the angle α increases from 0° to 360°. This arrangement has the disadvantage that for two parallel lines the angle α is not necessarily the same. If the lines are on opposite sides of the origin this angle differs by π. That is, if α is the angle for one of the lines, then

$\pi + \alpha$ (or $\alpha - \pi$) is the angle for the other. There are compensating advantages however which are believed to outweigh this one disadvantage. An alternative method of dealing with this problem can be found in several elementary texts.

205. The distance from a point to a line.—In solving the problem of finding the distance from a point to a line use is made of the normal form $x \cos \alpha + y \sin \alpha = p$ of the equation of the line. The essential points to be noted in connection with the development of the formula

$$d = \frac{Ax_1 + By_1 + C}{\pm \sqrt{A^2 + B^2}} \qquad [14]$$

giving the distance from (x_1, y_1) to the line $Ax + By + C = 0$ (see pages 44, 45) are,

(a) It is agreed arbitrarily that the distance from a point to a line shall be positive if the point is on the same side of the line as the origin; otherwise the distance is negative. Note that this is in conformity with the convention that the distance p from the origin to a line is positive for all lines.

(b) If two parallel lines are on the same sides of the origin, then their equations may be written $x \cos \alpha + y \sin \alpha = p$ (1) and $x \cos \alpha + y \sin \alpha = p'$ (2), α being the same in both equations.

If $P(x_1, y_1)$ is on (2), then $p - p'$ is the distance from $P(x_1, y_1)$ to (1), the distance agreeing in sign with the convention stated under (a) above.

(c) If two parallel lines are on opposite sides of the origin, their equations may be written $x \cos \alpha + y \sin \alpha = p$ (1) and $x \cos (\alpha + \pi) + y \sin (\alpha + \pi) = p'$ (2), α being the same in both equations.

If $P(x_1, y_1)$ is on (2) then $p + p'$ is the distance from $P(x_1, y_1)$ to (1). In this case the distance is always positive.

Note that this agrees with (a) above.

(d) If the parallel lines $Ax + By + C = 0$ and $Ax + By + C' = 0$ are on the same side of the origin, then C and C' have the

same sign, and if the lines are on opposite sides of the origin C and C' have opposite signs.

The use of these facts in the manner shown on pages 44, 45 gives the general formula for the distance from any point to any line. It is sometimes said that the only difficulty in analytic geometry appears in connection with the distance from a point to a line.

The equations of bisectors of angles formed by given lines are immediate consequences of the distance formula.

The angles formed by two lines whose slopes are given are found by using the formula $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.

Note that this formula gives two results, equal numerically but different in sign, depending upon the order in which α and β are taken since $\tan(\alpha - \beta) = -\tan(\beta - \alpha)$. The positive result is the tangent of the acute angles formed by the two intersecting lines and the negative result is the tangent of the obtuse angles.

206. Curves through the intersections of given curves.—The method used in §44 for finding the equation of a line through the intersection point of two given lines is obviously of very general application. That is, if $f_1(x, y) = 0$ (1) and $f^2(x, y) = 0$ (2), are the equations of any two curves, then

$$f_1(x, y) + kf_2(x, y) = 0 \qquad (3)$$

or if we prefer, as we do for some purposes,

$$hf_1(x, y) + kf_2(x, y) = 0 \qquad (3')$$

is the equation of a locus which passes through all intersection points of (1) and (2). The proof is exactly the same as that used in §44.

207. The second degree equation and its locus.—When the subject of analytic geometry is approached entirely from the algebraic side it is natural that the second degree equation and its locus should be studied immediately after the first degree equation.

How shall we attack the problem of determining what kind of curve may be represented by equations of the type

$$Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0 \qquad (1)$$

for various combinations of values of its Coefficients?

There are many possible ways of attacking this problem. The simplest way of making some kind of headway with it is to try special cases. The general equation of the circle would of course occur to any fairly good student when applying the distance formula [1] to the case when one point (a, b) is fixed and another (x, y) is permitted to vary. If we had no further advance information on this subject we should very likely proceed with the case $H = D = E = 0$, or $Ax^2 + By^2 + C = 0$, which of course would lead to the equations of the circle, ellipse, and hyperbola, with the origin as the center and the axes of the curves lying on the coördinate axes. It would not be particularly difficult to discover the "general appearance" of the ellipse and the hyperbola, but it would require somewhat of an inventive genius to discover the various special properties of these curves. Would the properties of the ellipse and the hyperbola stated in §§72, 86 be discovered? It is certainly not apparent at once that for the locus of $9x^2 + 16y^2 = 144$, for instance, there are two fixed points (the foci) such that the sum of the distances from these points to every point on the locus is exactly 8, nor is it apparent that for the locus of $9x^2 - 16y^2 = 144$ there are two points such that the difference of the distances from these points to every point of the locus is exactly 8. Equally concealed would be the properties of these curves which in chapters VI and VII of this book are used as the definitions of the curves. How would these be discovered? How is any discovery made? There is no general rule; there is no general formula by which inventions of any kind can be made. We simply have to hit upon them *somehow*. When a discovery is made it is added to the world's accumulat-

ing stock of information and from this new vantage ground we may reasonably hope to make other discoveries.

If now we were to add the special cases $B = H = E = 0$ and $A = H = D = 0$, we should have the parabola and hence all the conic sections.

But we should not yet have settled the question as to whether equation (1) might represent other curves. This question could be settled if we were in possession of the principal results obtained in Chapters XIII and VIII, namely that the term $2Hxy$ may be eliminated by rotation of the axes and the first degree terms by translating the axes (see §§160, 108, for exceptions to the last statement).

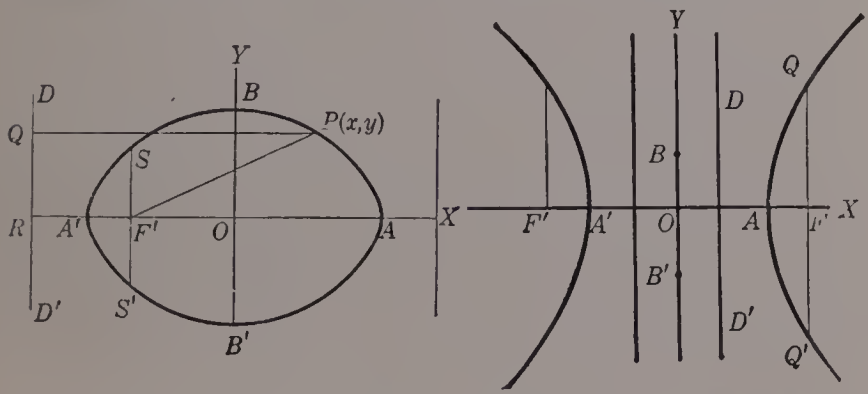
Chapters X, XI, XII, dealing with tangents, poles and polars, and diameters could now be developed exactly as in this book.

208. Special definitions of the conic sections.—Historically this subject was developed in a very different way from that suggested in §207. See pages 288, 292 of the Historical Sketch. The different curves and their important properties were discovered long before analytic geometry was invented. One of these properties was then seized upon as leading to simplicity of development when used as the definition of the curves. That is, a conic came to be defined as the locus of a point which moves so that its distances from a fixed point and a fixed line are in a constant ratio. We may remark that if we had no information on this subject it would be an achievement of no mean order to discover and prove that in case $Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0$ has a real locus then there is a fixed line (a directrix) and a fixed point (a focus) for which this property holds. A study of how this subject was unravelled step by step and the smoothness and simplicity of the theory finally obtained affords insight into the difficulties encountered in obtaining the vast heritage of information and methods for doing things which the human race now possesses.

209. Developing the equations of the ellipse and the hyperbola.—We note first that the definitions of the ellipse and the hyperbola given on pages 85 and 103 are identical except that in one case the eccentricity, e, is greater than unity, and in the other it is less than unity. Barring the slight discrepancy in notation that A' and F' are used in the case of the ellipse and A and F in the case of the hyperbola, the equations are developed in identical language until we reach the equation $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ (see pages 87 and 105), with the exception that in this book the work is slightly condensed in the case of the hyperbola. At this point note is taken of the difference in the values of e and we obtain at once the two equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

210. A problem in notation; choice of axes.—Smooth and suggestive notation is of great importance in mathematics and



it may be worth while to note why, in developing the equations of the ellipse and hyperbola, A and F is used in one case and A' and F' in the other. In the completed figures the notation is desired to be as shown in the figures.

That is, in each case A' and F' are the left vertex and focus and A and F the right vertex and focus. It is natural however,

to work from the left toward the right and hence the left part of the ellipse is used in the development and the right part of the hyperbola.

In each case the axes are chosen so that the equation will be in the simplest form, that is, so that the curves will be symmetrical with respect to both axes. It should be evident that a special choice of axes does not affect the properties of the curve and hence may be made solely with a view to simplicity.

211. Eccentricity; foci; directrices; latus rectum.—The expressions of the eccentricity in terms of a and b,

$$e = \frac{1}{a}\sqrt{a^2 - b^2}, \quad e = \frac{1}{a}\sqrt{a^2 + b^2}$$

give an obvious clue for remembering: $\frac{1}{a}\sqrt{a^2 - b^2}$, in case it has a real value at *all* is necessarily less than unity while $\frac{1}{a}\sqrt{a^2 + b^2}$ is real for all values of a and b and greater than unity.

The x-coördinates of the foci, $\pm ae$, are numerically less than a in case of the ellipse, and greater than a in case of the hyperbola. That is, the foci are between the vertices and the center in the ellipse, while the vertices are between the foci and the center in the hyperbola. The location of the directrices, $x = \pm \frac{a}{e}$, is likewise evident for the two curves. A glance at the figure in §210 reminds us of the location of the foci and directrices in these curves.

Putting $x = ae$, gives $y = \frac{b^2}{a}$ in both equations, (see pages 88, 105). Hence the latus rectum is $\frac{2b^2}{a}$ for both curves.

212. Parametric equation of tangents.—Solving the equation $y = mx + b$ simultaneously with each of the equations

$$x^2 + y^2 = r^2,\ y^2 = 4px,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

finding values of b which will make the expressions under the radical equal to zero, and substituting in $y = mx + b$ we have

$$y = mx \pm r\sqrt{1 + m^2}\ [18], \qquad y = mx + \frac{p}{m}\ [22],$$

$$y = mx \pm \sqrt{a^2m^2 + b^2}\ [30], \quad y = mx \pm \sqrt{a^2m^2 - b^2}\ [38].$$

Equations [18], [22], [30] represent tangents for all values of m, and [38] represents a tangent provided $a^2m^2 - b^2$ is positive. It is at once apparent that the numerical value of the slope of a tangent to the hyperbola is always greater than $\frac{b}{a}$. In case $m = \frac{b}{a}$, the lines are asymptotes. The asymptotes are sometimes said to be tangent to the curve at infinity. This means simply that as the point (x, y) recedes indefinitely along a branch of the curve, the tangent approaches the asymptote as a limiting position. We are not considering here possible artificial curves for which this statement is not necessarily true.

213. Equation of tangent to conic at a point on it.—The method used in §§51, 61, 76, 90 for finding the equation of the tangent at a given point (x_1, y_1) on the various conics is exactly the same for all these curves. That this is necessarily so is shown in §124 where the equation of the tangent is developed using the more general equation $Ax^2 + By^2 + 2Dx + 2Ey + C = 0$, which includes the others as special cases.

The forms of the equations of the tangents should be noted, viz.:

$$x_1x + y_1y = r^2\ [19], \qquad y_1y = 2p(x + x_1)\ [23],$$

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1\ [32], \qquad \frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1\ [39].$$

$$Ax_1x + By_1y + D(x + x_1) + E(y + y_1) + C = 0 \qquad [46]$$

Each of these is obtained by replacing x^2, y^2, x, y in the equation of the curve by x_1x, y_1y, $\frac{x\,x_1}{2}$, $\frac{y\,y_1}{2}$ respectively. Obviously [19], [23], [32], [39], are special cases of [46].

214. Miscellaneous properties of the conics.—The formulas giving the tangents of the angles formed by the focal radii and the normal and tangent to the ellipse and hyperbola (§§77, 91) are found in a closely similar manner for the two curves, and it will be illuminating to write out these developments side by side.

The optical property of the parabola (§64) may be regarded as the limiting case of the optical property of the ellipse (§78) where one focus is allowed to recede indefinitely toward the right while the left vertex and focus are held fixed. Compare §198.

The properties of the ellipse and hyperbola stated in §§72, 86 should be developed simultaneously in review. It is interesting to develop the equations of the curves using these properties as the definitions.

215. The general equation of the second degree.—All the properties of the conic sections have been (or may be) developed from their special equations. When we turn to the general equation of the second degree the real point at issue is as to the kinds of curves represented by this equation. In this book this problem is solved by showing that by rotating and translating the axes, this general equation may be reduced to one which we already know represents a conic section if it has a real locus. Why did we study the individual conics first, in some cases defining them by stating a property that is not at all apparent and with which we could become acquainted only after a fairly exhaustive study of these curves? Why did we not do what some would call the natural thing by attacking the general second degree equation when we had finished our study of the first degree equation? To an extent the choice may be a matter of taste. The alternative of the course adopted in this book would certainly lead to very serious difficulty (see §207 for instance) and if we tried it we should probably be driven to study special cases separately as we have done. Some topics are no doubt on the border line and we have treated

them both ways. Thus tangents were treated separately for the different conics and again once for all for the more general form $Ax^2 + By^2 + 2Dx + 2Ey + C = 0$. On the other hand, more advanced topics such as diameters and poles and polars were considered for the more general case only. Obviously these might have been treated separately for the different curves as were the tangents, had we so desired.

216. Curves other than those of the first and second degree.—In an elementary course curves not of the first or second degree must necessarily be treated piecemeal. The general third degree curve, for instance, is far beyond our scope. The higher plane curves considered in this book all arise from special definitions. Our problem is to translate these definitions into equations which may be algebraic or trigonometric, or in some cases exponential, and to study their properties by means of these equations. In some cases such as $r = \sin n\theta$ or $r = \cos n\theta$, the definition is given initially in the form of an equation.

217. Algebra used in this course; the invention of analytic geometry.—Since by far the greater part of this book is concerned with loci represented by equations of the first and second degree it follows that the algebra involved is nearly all confined to linear and quadratic equations, though in some cases problems bring in equations of higher degree. See, for instance, Ex. 2, page 215. The solution of the quadratic, and the relations between the coefficients and the roots of a quadratic, are used particularly frequently. See pages 60, 71, 76, 83, 91, 92, 108, 109, 118.

In the history of mathematics no chapter is more interesting than that dealing with analytic geometry, for nowhere does the significance of a method stand out more clearly. As shown in the historical sketch, nearly every fact about pure geometry stated in an elementary text such as this one was known before the method of analytic geometry was invented. Why did the Greeks fail to invent analytic geometry? Why is any invention

delayed as long as it is? For one thing any far-reaching invention usually makes use of other inventions and must necessarily wait on them. The Greeks did not have a workable algebra, which, as we know, is one complete aspect of analytic geometry. Again the Greeks were so fully committed to the method of Euclidian geometry that to them a wide departure from it would seem impossible. Magnitudes were represented by the lengths of lines and the areas of rectangles; problems which we would solve by means of ordinary algebraic equations were solved by them by means of geometrical construction; instead of solving geometric problems by means of algebra as we do, they solved algebraic problems by means of geometry.

After the Greeks came the long "dark ages," especially as regards mathematics. When the light broke again—when the "new birth," the "renaissance," came—the first work in mathematics was in algebra. A notation was perfected and algebraic equations were solved, notably the cubic and the biquadratic. Then it was that Descartes and Fermat (see pages 294, 296) hit upon the scheme of representing the relation of the coördinates of a point by means of an algebraic equation. The ease with which the problem of Pappus (see page 290) was solved by the new method illustrates admirably its superior power, for the Greeks with their transcendent genius failed to solve it by the older method. The latter was first achieved by Sir Isaac Newton who, after analytic geometry was invented, solved the problem by the method of Euclidian geometry. But there were few imitators of Newton in this respect. Since the time of Descartes important problems in geometry have been solved almost exclusively by using the methods which were first published by him, that is, by the methods which we have studied in this book. Other methods such as those of the calculus have of course been used, but analytic geometry is used in the calculus as elementary arithmetic is used in algebra.

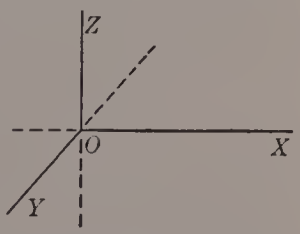
CHAPTER XVIII

SOLID ANALYTIC GEOMETRY

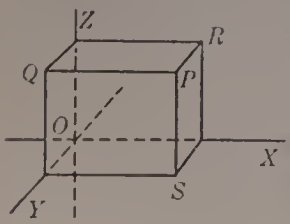
Solid analytic geometry, or analytic geometry of three dimensions, is closely analogous to plane analytic geometry. Instead of two coördinate axes, and two coördinates, x and y, determining a point (x, y), we have three coördinate axes and three coördinate planes, a point (x, y, z) being determined by the three coördinates x, y, and z.

In this chapter we will consider briefly certain parts of solid geometry of which use is often made in elementary calculus.

218. Three dimensional coördinates.—In the figure, OX, OY, and OZ are three lines, each one perpendicular to each of the other two. That is, the line OX is perpendicular to both OY and OZ and hence to the plane YOZ. Similarly OY is perpendicular to the plane ZOX, and OZ is perpendicular to the plane XOY. The lines OX, OY, and OZ are called the **coördinate axes,** and the planes XOY, YOZ, and ZOX, called respectively the xy-plane, the yz-plane, and zx-plane, are the **coördinate planes.**



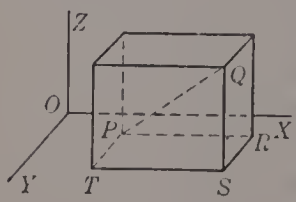
The distances $QP = x$, $RP = y$, $SP = z$ from the three coördinate planes to a point P are the **coördinates** of the point $P(x, y, z)$. The coördinates of a point are positive or negative, depending upon the side of each coördinate plane on which the point lies. The signs are usually arranged so that the letters X, Y, and Z, as shown in the figure, are on the positive side of the yz-, the zx-, and the xy-plane, respectively.



The coördinate planes divide the space about the point O into eight parts called **octants**, which may be numbered as are the quadrants in plane geometry. In octants I to IV the z-coördinate is positive and the x- and y-coördinates have in order the signs $++$, $-+$, $--$, $+-$ as in plane geometry, while in octants V to VIII the z-coördinate is negative and the x- and y-coördinates are repeated in the same order. The octants may therefore be characterized in their order by the following combinations of signs:

I: $+++$. II: $-++$. III: $---+$. IV: $+-+$.
V: $++-$. VI: $-+-$. VII: $---$. VIII: $+--$.

219. The distance between two points.—Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points in space. As shown in the figure, construct planes through P and Q parallel to the coördinate planes forming a rectangular parallelopiped with P and Q as opposite vertices. Then, using the notation of the figure, PR, RS, and SQ are equal in length to the absolute values of $x_1 - x_2$, $y_1 - y_2$, and $z_1 - z_2$ respectively.



Hence $\mathbf{PQ} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ [52]

Compare §9.

220. Point dividing a segment in a given ratio.—Let $R(x, y, z)$ be a point on the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. If R is the middle point of this segment, then proceeding as in §10 we have,

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}. \qquad [53]$$

If $PR : RQ = m : n$, then proceeding as in §11,

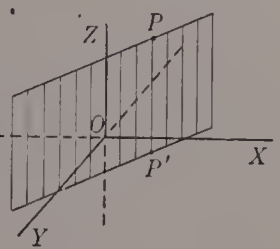
$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}, \quad z = \frac{mz_2 + n\,z_1}{m + n}. \qquad [54]$$

Obviously R may be any point on the line PQ (except Q), the segment PQ being divided either internally or externally.

221. Planes and lines perpendicular to coördinate planes.— If we confine ourselves to the x-axis, then $x = a$ represents a point on that axis. In ordinary plane analytic geometry, which is restricted to a single plane, the equation $x = a$ represents a line perpendicular to the x-axis, namely the line all of whose points have the abscissa a. In solid analytic geometry, which deals with three-dimensional space, $x = a$ represents a plane perpendicular to the x-axis, namely the plane all of whose points are a distance a from the yz-plane. Similarly $y = b$ represents a plane perpendicular to the y-axis and $z = c$ represents a plane perpendicular to the z-axis.

Consider a plane perpendicular to the xy-plane and meeting it in the line

$$Ax + By + C = 0. \qquad (1)$$



Let $P(x_1, y_1, z_1)$ be a point on this plane. The perpendicular from P to the xy-plane meets it in a point P' on (1). Moreover, this perpendicular is parallel to each of the other two coördinate planes and hence all points on it are the same distance from each of these planes. Therefore x_1, y_1, satisfy (1), and we have that for all points in this plane the relation between the x- and y-coördinates is that stated in (1).

Hence in solid geometry (1) is the equation of the plane perpendicular to the xy-plane and meeting it in the line $Ax + By + C = 0$ when this is regarded as an equation in plane geometry. Evidently any plane perpendicular to the xy-plane may be represented by an equation of this type.

It is now clear that while in plane geometry two equations $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ determine a point (x_1, y_1) namely the point in which the lines meet, in solid geometry these two equations determine a line, namely the line perpendicular to the xy-plane at the point (x_1, y_1).

222. The equation of a plane.—A plane has the property that a line through any two of its points lies wholly in the plane. This statement holds for every plane and for no other figure except a single straight line and except also the whole of space. If these are excluded, the property just stated may be used as the definition of a plane.

We will now prove that

$$Ax + By + Cz + D = 0 \qquad [55]$$

represents a plane.

Let $Q(x_1, y_1, z_1)$ and $R(x_2, y_2, z_2)$ be any two points on the locus of this equation and let $P(x, y, z)$ be any point on the line QR.

Then $$Ax_1 + By_1 + Cz_1 + D = 0 \qquad (1)$$

and $$Ax_2 + By_2 + Cz_2 + D = 0. \qquad (2)$$

By §220 there are numbers m and n such that

$$x = \frac{mx_2 + nx_1}{m + n},\ y = \frac{my_2 + ny_1}{m + n},\ z = \frac{mz_2 + nz_1}{m + n}. \qquad (3)$$

Substituting these values of x, y, z in the left member of [55] it may be reduced to

$$\frac{n(Ax_1 + By_1 + Cz_1 + D) + m(Ax_2 + By_2 + Cz_2 + D)}{m + n}.$$

But by (1) and (2) the expressions in parentheses are both equal to zero and hence the whole expression equals zero. That is, x, y, z satisfy [55] and the point (x, y, z) is on the locus of [55].

It is left for the student to verify that the locus of [55] is not a single straight line and that it is not the whole of space. Hence the locus is a plane.

EXERCISE

Show without using the proposition proved in §222 that the locus of $Ax + By + Cz + D = 0$ cuts each of the coördinate planes in a straight line. Does this result prove that the equation represents a plane? Does it verify this proposition?

223. Forms of the equation of a plane.—Obviously

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \qquad [56]$$

is the equation of a plane through the point (x_1, y_1, z_1) for all values of A, B, and C. Later (§225) we shall consider the significance of the constants A, B, C.

The equation of the plane through the points $A(a, 0, 0)$, $B(0, b, 0)$, and $C(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad [57]$$

as is evident on substituting the coördinates of the points in the equation. Hence a, b, c are the intercepts on the axes of this plane. This equation is the intercept form of the equation of the plane. Compare §31.

The equation corresponding to the "normal" equation of the line in plane geometry is obtained as follows:

Let $OP = p$ be the perpendicular from the origin upon the plane [57]. Denote the angles AOP, BOP, and COP by α, β, and γ, and as above let $OA = a$, $OB = b$, $OC = c$. Since the triangles AOP, BOP, COP are right angled at P,

$$\cos \alpha = \frac{p}{a}, \cos \beta = \frac{p}{b}, \cos \gamma = \frac{p}{c}.$$

Solving these equations for a, b, and c and substituting in [57] we have

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p \qquad [58]$$

which is the normal form of the equation of the plane.

To compare with the corresponding equation $x \cos \alpha + y \sin \alpha = p$ in §32, suppose that P is in the xy-plane. Then $\sin \alpha = \cos \beta$ and $\cos \gamma = 0$. Hence [58] reduces to the normal form of the equation of a straight line.

The equation of the plane through three points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ may be found by substituting these coördinates in

$$Ax + By + Cz + D = 0$$

and solving the three equations for $\frac{B}{A}, \frac{C}{A}, \frac{D}{A}$.

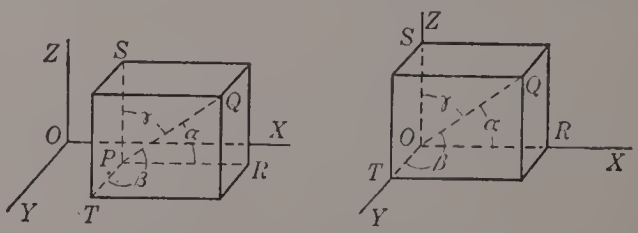
The equation of the plane through three points may also be written directly in the determinant form.

That is,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

is an equation of the first degree in x, y, z and is satisfied by (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). See §§185, 196, 249.

224. The direction cosines of a line.—Let the line PQ in the first figure be any given line. Through O (see the second figure) construct a line parallel to PQ and let the angles which this line makes with the coördinate axes be α, β, and γ.



Then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **direction cosines** of the line PQ (and of every line parallel to it).

These direction cosines we will denote by l, m, n.

Clearly the angles α, β, γ are equal to the angles RPQ, TPQ, and SPQ shown in the first figure. Denoting PQ by d we have

$$l = \frac{PR}{d},\ m = \frac{PT}{d},\ n = \frac{PS}{d}.$$

Letting P and Q be the points (x_1, y_1, z_1) and (x_2, y_2, z_2) and using absolute values we have $PR = x_1 - x_2$, $PT = y_1 - y_2$, $PS = z_1 - z_2$. Hence using the value of $d = PQ$ found in §219,

$$l^2 + m^2 + n^2 = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = 1.$$

That is, the sum of the squares of the direction cosines of a line is unity.

225. The direction cosines of the normal to a plane.—Consider the general equation of a plane

$$Ax + By + Cz + D = 0. \tag{1}$$

Reducing (1) to the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (2)

and noting that the direction cosines of a normal to (2) are $\frac{p}{a}, \frac{p}{b}, \frac{p}{c}$ (see §223), we find that the direction cosines of the normal to (1) are

$$l = -p\frac{A}{D},\ m = -p\frac{B}{D},\ n = -p\frac{C}{D}.$$

That is, $l : m : n = A : B : C$.

Hence the direction cosines of the normal on the plane (1) are proportional to the coefficients of x, y, and z in that equation.

It follows that in the equation

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

the direction cosines of the normal on the plane are proportional to $A : B : C$. Hence we have that

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0 \tag{59}$$

is the equation of the plane through (x_1, y_1, z_1), the direction cosines of the normal to which are l, m, n.

Denoting $\sqrt{A^2 + B^2 + C^2}$ by R, the actual values of the direction cosines of the normal on the plane $Ax + By + Cz + D = 0$ are, $l = \frac{A}{R},\ m = \frac{B}{R},\ n = \frac{C}{R}$.

This argument does not hold when the plane passes through the origin; that is, when the equation of the plane is in the form $Ax + By + Cz = 0$ (3). In this case consider the plane $Ax + By + Cz + D = 0$ (4) where $D \neq 0$. Now show that (3) and (4) are parallel and hence that these planes have the same normals. Then show that $l = \frac{A}{R},\ m = \frac{B}{R},\ n = \frac{C}{R}$ for the plane (3).

226. The equation of a line through a given point with given direction cosines.—Let $P(x_1, y_1, z_1)$ be the given point and l, m, and n, the given direction cosines. Let $Q(x, y, z)$ be any point on the line. It then follows as in §224 that

$$l = \frac{x - x_1}{d}, \; m = \frac{y - y_1}{d}, \; n = \frac{z - z_1}{d}.$$

Hence,
$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad [60]$$

Comparing this equation with

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0 \text{ (see §225)}, \qquad (2)$$

we note that the line represented by (1) is normal to plane (2).

227. Line determined by two planes.—If l, m, and n, are the direction cosines of the line determined by the two planes

$$A_1x + B_1y + C_1z + D_1 = 0$$

and
$$A_2x + B_2y + C_2z + D_2 = 0,$$

then it may be shown that

$$l : m : n = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} : \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} : \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

If $\sqrt{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}^2 + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}^2 + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}^2}$ is denoted by R. then

$$l = \frac{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}}{R}, \quad m = \frac{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}}{R}, \quad n = \frac{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}{R}.$$

228. The angle between two lines.—The direction of a line is obviously determined by its direction cosines. Hence we are led to try to express the angle formed by two lines in terms of their direction cosines.

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two lines through the origin; let θ be the angle formed by these lines; and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points one on each of the given lines. Denote the distances OP, OQ, and PQ by d_1, d_2, and d respectively.

Then $d^2 = d_1^2 + d_2^2 - 2d_1d_2 \cos\theta$ (see §253) (1)

Also, $x_1 = l_1d_1$, $y_1 = m_1d_1$, $z_1 = n_1d_1$, and
$x_2 = l_2d_2$, $y_2 = m_2d_2$, $z_2 = n_2d_2$.

Then $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$
$= (l_1d_1 - l_2d_2)^2 + (m_1d_1 - m_2d_2)^2 + (n_1d_1 - n_2d_2)^2$.

Expanding and noting that $l_1^2 + m_1^2 + n_1^2 = 1$ and $l_2^2 + m_2^2 + n_2^2 = 1$, this reduces to

$$d^2 = d_1^2 + d_2^2 - 2d_1d_2\,(l_1l_2 + m_1m_2 + n_1n_2). \qquad (2)$$

From (1) and (2) it follows that

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \qquad [61]$$

229. The angle between two planes.—The angle between two planes is equal to the angle between the normals to these planes.

Consider two planes

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0.$$

As in §225 denote $\sqrt{A_1^2 + B_1^2 + C_1^2}$ by R_1 and $\sqrt{A_2^2 + B_2^2 + C_2^2}$ by R_2.

The direction cosines (see §225) of these planes are

$$l_1 = \frac{A_1}{R_1},\ m_1 = \frac{B_1}{R_1},\ n_1 = \frac{C_1}{R_1}$$

and

$$l_2 = \frac{A_2}{R_2},\ m_2 = \frac{B_2}{R_2},\ n_2 = \frac{C_2}{R_2}.$$

Then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$$= \frac{A_1A_2 + B_1B_2 + C_1C_2}{R_1R_2}$$

$$= \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\cdot\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

That is,

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\cdot\sqrt{A_2^2 + B_2^2 + C_2^2}} \qquad [62]$$

where θ is the angle between the two planes.

PROBLEMS ON STRAIGHT LINES AND PLANES

1. Find the intercepts on the axes of the plane $2x + 3y - 5z = 30$.

2. Find the equation of the plane whose intercepts on the axes are 3, -1, 4.

3. Transform the equation of Ex. 1 into the normal form.

4. Find the equation of a plane through the point $(4, -2, 6)$ if the direction cosines of the normal upon it are proportional to $-1:4:2$.

5. Find the coördinates of the point of intersection of the planes $x + 2y = 8$, $x - 2y + z = 0$, and $x - y + z = 3$.

6. Find the equation of the plane through the intersection of the planes $2x + y - z = 4$ and $x - y + 2z = 0$ which passes through the point $(0, -1, 1)$.

SUGGESTION: Use the method of §44.

7. Find the equations of the line through the point $(4, -1, -2)$ which is perpendicular to the plane $2x - 3y + z = 12$.

8. Find the direction cosines of the line passing through the points $(2, -1, 6)$, $(-1, 2, 3)$.

9. Find the cosines of the angles of the triangle whose vertices are $(2, -1, 6)$, $(-1, 2, 3)$, $(0, 0, 0)$.

10. Find the cosine of the angle formed by the planes $8x - 2y + 3z = 16$, $2x + 4y - z = 8$.

11. Prove that the medians of the triangle in Ex. 9 meet in a point.

12. Find the equation of the locus of points which are equidistant from the points $(-1, 6, 5)$, $(2, -1, 2)$.

13. Find the direction cosines of the line $4x - 2y - z = 6$, $2x + 5y - 5z = 12$.

SUGGESTION: First find two points on the line by giving z arbitrary values and finding the corresponding values of x and y, or find one point.

14. Find the equation of a plane through the point $(-7, 2, 1)$ which is perpendicular to the line of Ex. 13.

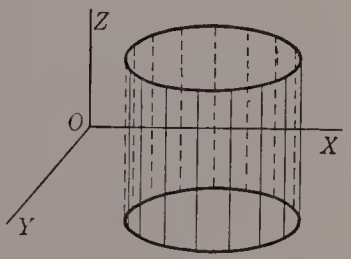
15. Find equations of the line through $(-7, 2, 1)$ which is parallel to the line of Ex. 13.

16. Find the cosine of the angle formed by the planes $3x - 2y + z - 6 = 0$, $x + 5y - 3z + 4 = 0$.

17. State a condition that the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ shall be parallel. Also state a condition that these planes shall be perpendicular to each other.

230. Cylindrical surfaces.—In plane geometry an equation of the type $f(x, y) = 0$ represents a curve provided it has a real locus. Note that a straight line is here included in the class of loci called curves. In solid geometry such an equation represents what is called a **cylindrical surface** which is normal to the xy-plane. This surface is the locus of all straight lines which are perpendicular to the xy-plane and which meet it in the curve $f(x, y) = 0$.

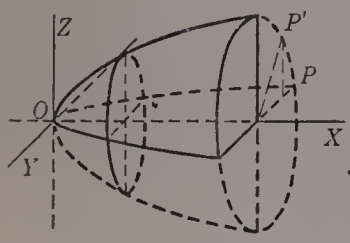
A cylindrical surface may be an ordinary closed cylindrical surface such as $x^2 + y^2 = r^2$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or it may be "open" such as $y^2 = 4px$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $y = x^3$, $y = \sin x$, etc.,



or even a plane such as $Ax + By + C = 0$ (see §221). The figure shows a circular cylinder whose equation is of the form $(x - a)^2 + (y - b)^2 = r^2$.

231. Surfaces of revolution.—A surface obtained by revolving a curve about a fixed line as an axis is called a **surface of revolution.**

Thus a sphere is obtained by revolving a circle about a fixed diameter.



If in $f(x, y) = 0$, y^2 is replaced by $y^2 + z^2$ (or y by $\sqrt{y^2 + z^2}$) the resulting equation represents a surface obtained by revolving the curve $f(x, y) = 0$ about the x-axis.

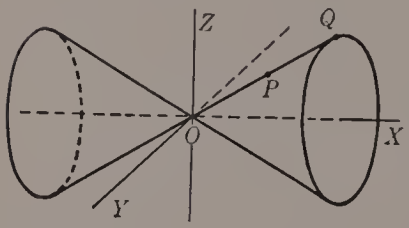
For, let $P(x_1, y_1)$ be a point on $f(x, y) = 0$.

Then the coördinates of any point P' on the circle described by P in revolving the curve about the x-axis will be at a fixed distance $y_1 = \sqrt{y^2 + z^2}$ from the axis and hence $f(x, \sqrt{y^2 + z^2})$ will be satisfied by the coördinates of every point on this circle.

232. Conical surfaces.—Consider the equation

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 0 \qquad [63]$$

Obviously the origin is a point on the locus of this equation since the equation is satisfied by 0, 0, 0. Let $P(x_1, y_1, z_1)$ be another point on the locus and let $Q(x_2, y_2, z_2)$ be any point on the line OP. Then by §220, noting that $O(0, 0, 0)$ is one of the points, there are two numbers m and n such that



$$x_2 = \frac{mx_1}{m+n},\ y_2 = \frac{my_1}{m+n},\ z_2 = \frac{mz_1}{m+n}. \qquad (1)$$

Since by hypothesis x_1, y_1, z_1 satisfy [63] we find by substituting from (1) in [63] that [63] is satisfied by x_2, y_2, z_2.

That is, a line through the origin which meets the locus of [63] in any other point lies wholly in this locus. This is the characteristic property of what is called a conical surface. The formal definition of such a surface is:

If on a surface there is a point V such that every line through V which contains another point of the surface lies wholly in the surface, then the surface is a **conical surface.** The point V is called the **vertex** of the surface[1].

It is now evident that, with the exception noted below, the argument used above applies for any combination of signs in

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0.$$

We note that when all the signs are positive the only point on the locus is the origin. Hence in this case the equation does not represent a conical surface.

[1] Note that this definition of conical surface includes a plane among these surfaces. In the case of the plane every point is a vertex.

It is easily seen that in

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 0, \quad \text{or} \quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2},$$

every section of the surface made by a plane perpendicular to the x-axis, $x = x_1$, is an ellipse whose equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x_1^2}{a^2}, \quad \text{or} \quad \frac{y^2}{\frac{b^2 x_1^2}{a^2}} + \frac{z^2}{\frac{c^2 x_1^2}{a^2}} = 1.$$

That is, the semiaxes of the ellipse are $\frac{bx_1}{a}$, and $\frac{cx_1}{a}$.

EXERCISES

1. Discuss the character of the intersections of planes $y = y_1$, and $z = z_1$ with the surface $\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 0$.

2. Discuss the character of the intersections of planes $x = x_1$, $y = y_1$, $z = z_1$ with the surface $\frac{y^2}{b^2} - \frac{z^2}{c^2} - \frac{x^2}{a^2} = 0$ and also with the surface $\frac{y^2}{b^2} - \frac{z^2}{c^2} + \frac{x^2}{a^2} = 0$.

233. The sphere.—From the definition of a sphere and the distance formula of §219 we have that the equation of a sphere with center (x_1, y_1, z_1) and radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2. \tag{1}$$

If the center is at the origin this equation reduces to

$$x^2 + y^2 + z^2 = r^2. \tag{2}$$

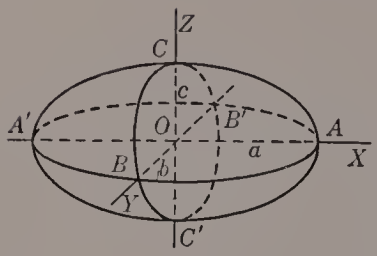
EXERCISES

1. Find equation (2) of this section by regarding the sphere as developed by revolving the circle $x^2 + y^2 = r^2$ about the x-axis and using the method of §231.

2. Show that a plane perpendicular to one of the coördinate axes cuts the sphere in a circle. Why is the restriction made here with respect to the cutting plane?

234. The ellipsoid.—Consider the locus of the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$



Answer the following questions.

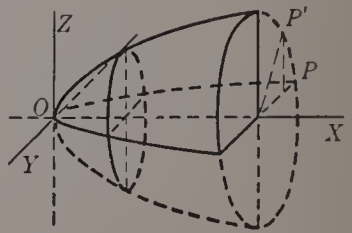
(a) What are the limits of the possible values of x in this equation? of y? of z?

(b) In what kind of curve does a plane perpendicular to one of the coördinate axes cut this surface?

(c) If $b=c$, in what kind of curve does $x=x_1$ cut this surface? In this case is this a surface of revolution? (Apply §231.)

235. Paraboloids.—Consider the locus of the equation

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=4px.$$

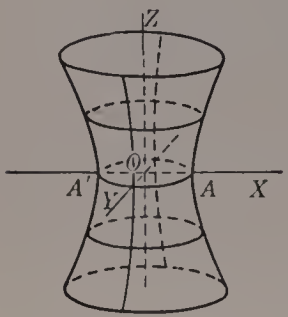


Assume p positive; then answer the following questions.

(a) What are the limits of the possible values of x in this equation? of y? of z?

(b) In what kind of curve does a plane $x=x_1$ cut this surface?

(c) In what kinds of curves do planes $y=y_1$ and $z=z_1$ cut this surface?



(d) Discuss this surface for the case when $b=c$.

236. Hyperboloids.—Consider the locus of equations of the type

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1$$

not including the case when all signs are positive. State and answer questions on this equation similar to those in §235.

CHAPTER XIX

THEOREMS AND FORMULAS FROM ALGEBRA AND TRIGONOMETRY

In this chapter are collected for convenient reference some of the results obtained in more elementary courses. No proofs are given, the object being merely to refresh the student's memory. For discussion and proofs, and possibly for additional theorems and formulas, there should be at hand texts on plane geometry, trigonometry, and college algebra. We will now give in order theorems from these subjects.

Plane Geometry

237. The Pythagorean proposition.—If a, b, c are the sides of a right triangle, then

$$c^2 = a^2 + b^2.$$

238. Similar triangles.—If ABC and $A'B'C'$ are similar triangles, then

$$a : a' = b : b' = c : c',$$

a, b, c and a', b', c' being corresponding sides of the triangles.

239. A mean proportional between two segments.—If C is any point on a circle of which AB is a diameter, and if CD is a perpendicular from C on AB, then CD is a mean proportional between AD and DB.

That is, $$\overline{CD}^2 = AD \cdot DB.$$

This theorem may also be stated as follows:

If the triangle ABC is right angled at C and if CD is the altitude on AB, then CD is a mean proportional between AD and DB.

Algebra

240. Solution of quadratic equations.—The solution of the general quadratic $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

By means of this formula the solution of any quadratic may be written at once. In the case of numerical coefficients approximate values may be obtained by using the table on page 298, to find square roots.

241. Relations between coefficients of a quadratic and its roots.—In the general quadratic $ax^2 + bx + c = 0$ the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. Clearly, if the equation is in the form $x^2 + px + q = 0$, then q is the product of the roots and $-p$ is their sum.

In some problems (see Example 2, page 118) one-half the sum of the roots is required. This may be found at once without solving the equation.

242. Discriminant of the quadratic function $ax^2 + bx + c$.—In the solution $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ of the equation $ax^2 + bx + c = 0$, $b^2 - 4ac$ is the **discriminant.** If the discriminant is equal to zero, then the two roots of the equation are equal. That is, the equation has only one root. Under this condition $ax^2 + bx + c$ is a perfect square, its square root being $\sqrt{a}\,x + \sqrt{c}$ (see Example 3, page 22; also pages 60, 76, etc., and numerous examples and discussions where tangents are found).

243. Discriminant of the quadratic function $Ax^2 + By^2 + 2Dx + 2Ey + C$.—If $Ax^2 + By^2 + 2Dx + 2Ey + C$ can be factored into two factors which are rational in x and y, then solving $Ax^2 + By^2 + 2Dx + 2Ey + C = 0$ for x (or y as we may choose) must give a result in which y (or x) is not under the radical sign.

That is, the expression under the radical must be a square.

But $x = \dfrac{-D \pm \sqrt{D^2 - ABy^2 - 2AEy - AC}}{A}$.

Then by §242, $4A^2E^2 + 4AB(D^2 - AC) = 0$,

or $$AE^2 + BD^2 - ABC = 0.$$

The expression $AE^2 + BD^2 - ABC$ is the discriminant of $Ax^2 + By^2 + 2Dx + 2Ey + C$. Note that the factors may be imaginary *in the constants* even if the discriminant equals zero. Thus B and E may be zero and $D^2 - AC$ negative. The discriminant being zero simply insures that the given expression can be broken up into linear factors which are real *in the variables*.

244. Discriminant of $Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C$.—Proceeding exactly as in §243 we find that the given expression may be factored into linear factors in x and y if $AE^2 + BD^2 + CH^2 - ABC - 2DEH = 0$.

This expression is the discriminant of the most general quadratic function of x and y when in the form given in the title of this article. Note that the discriminant just given reduces to that of §243 when $H = 0$. The discriminant is denoted by Δ.

245. Other forms of the discriminant.—In the process of obtaining the discriminant in §244 its first form is

$$\Delta = \frac{1}{A}\left[(DH - AE)^2 - (H^2 - AB)(D^2 - AC)\right].$$

This form is often more convenient than the simplified form given in §244.

By expanding the determinant it is shown that

$$\begin{vmatrix} A & H & D \\ H & B & E \\ D & E & C \end{vmatrix} = -\Delta.$$

This form is sometimes convenient.

Using this notation we say that $\Delta = 0$ is the condition that the equation $Ax^2 + By^2 + 2Hxy + 2Dx + 2Ey + C = 0$ shall represent two straight lines if its locus is real.

246. Theorem on divisibility.—The following theorem is easily proved. If $x = r_1$ is a root of the equation

$$p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0,$$

then $x - r_1$ is a factor of the polynomial $p(x)$.

That is,

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = (x - r_1)(a_0x^{n-1} + a'_1x^{n-2} + \ldots + a'_{n-2}x + a'_{n-1}).$$

247. Determinants of the third order.—The square array of numbers

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - b_1a_2c_3 - a_1c_2b_3$$

is a determinant and its value is defined as that given in the expression in the right member of the equation.

248. Expansion of any determinant.—Consider the fourth order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Using the general definition of a determinant it turns out that the value of this determinant is

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

A determinant of any order may be evaluated in a similar manner. For the evaluation of a sixth order determinant see page 221 of this book.

249. Zero determinants.—Using the definition of a determinant implied in §§247, 248, it turns out that if two rows (or two columns) of a determinant are identical, the determinant is equal to zero. This property of determinants makes it particularly easy to write in the determinant form the equation of a straight line (§185), a circle (§187), or any conic (§196).

Trigonometry

250. Definitions of trigonometric functions.—If O is the origin in a rectangular system of coördinates, A a point on the positive part of the x-axis, $P(x, y)$ any point in the plane, and p the distance from O to P, then, denoting the angle AOP by θ, the trigonometric functions are defined as follows:

$$\sin\theta = \frac{y}{p} \qquad \tan\theta = \frac{y}{x} \qquad \sec\theta = \frac{p}{x}$$

$$\cos\theta = \frac{x}{p} \qquad \cot\theta = \frac{x}{y} \qquad \csc\theta = \frac{p}{y}.$$

In these definitions p, the distance from O to P, is regarded as positive for all positions of P.

251. Relations between trigonometric functions.—The following relations are used in this book.

(1) $\sin^2\theta + \cos^2\theta = 1$ (5) $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$

(2) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (6) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

(3) $\sin\theta = \sin(\pi - \theta)$ (7) $\sin\theta = -\sin(\pi + \theta)$

(4) $\cos\theta = -\cos(\pi - \theta)$ (8) $\cos\theta = -\cos(\pi + \theta)$

Of these formulas (1) is a direct consequence of the definitions of sine and cosine and the distance formula [1], §9; (2) is a direct consequence of the definition of sine, cosine, and tangent; (3) – (8) can be verified by referring to the graphs of $\sin x$ and $\cos x$ on page 194.

252. Addition formulas in trigonometry.—The fundamental addition formulas are:

(1) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
(2) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
(3) $\cos(A + B) = \cos A \cos B - \sin A \sin B$
(4) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

From these follow directly

(5) $\sin 2A = 2 \sin A \cos A$, (6) $\cos 2A = \cos^2 A - \sin^2 A$, and also

$$(7)\ \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$(8)\ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

If $A = B \pm \pi/2$, that is, if A and B differ by $\pi/2$ then

$$(9)\ \tan A = -\frac{1}{\tan B}, \text{ or } \tan A \cdot \tan B = -1.$$

253. Cosine formula.—In any triangle ABC whose sides are a, b, c, we have

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

254. Sines and cosines of special angles.—The following table is of use in constructing certain curves.

Angle.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Sin.	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
Cos.	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{3}$	-1

255. Projections.—(1) If a segment AB makes an angle θ with the x-axis and if $A'B'$ is its projection on the x-axis and $A''B''$ is its projection on the y-axis, then $A'B' = AB \cos \theta$ and $A''B'' = AB \sin \theta$.

(2) If AB and BC are any two segments connecting A and C, then the sum of the projections of AB and BC on either axis is equal to the projection of AC on this axis.

That is, x-proj $AB + x$-proj $BC = x$-proj AC

and y-proj $AB + y$-proj $BC = y$-proj AC.

HISTORICAL SKETCH

The publication in 1637 of *La Géométrie* by René Descartes marks definitely the date of Analytic Geometry in the history of mathematics. Like other great advances in mathematics analytic geometry is essentially a method, which may be used either in a simpler and more effective formulation of facts known before its invention, or in investigations that result in widening the boundaries of our knowledge. The tremendous genius of the Greeks enabled them to discover properties of geometric figures, which, while relatively simple when modern methods are used, are astounding when we consider the theoretical equipment with which they worked. It is easy to shoot grizzly bears with a high-powered rifle but a very different matter to kill them with a hunting knife. Nearly all the properties of lines and curves studied in an elementary text such as the present were known before the time of Descartes and nearly all this knowledge was due to the Greeks.

The older method for describing curves and studying their properties was essentially that of the elementary (Euclidean) geometry which we study in our high schools, and especially the method which we use in plane geometry in studying "loci." The more complicated relations were described by using proportion. There was not even a remote approach to the modern method of describing curves by indeterminate equations in two (or more) variables and then investigating the properties of the curves by algebraic manipulation of these equations.

We will begin this historical sketch with a résumé of the historical development of that part of geometry which is treated in this book by the method of analysis. Early in Greek

history three problems in geometry became famous, and a surprisingly large share of the work done in antiquity in what was then higher mathematics found its inception in connection with efforts to solve these problems. It is very likely true that one of the so-called higher plane curves, the quadratrix (§172) was described before the discovery of any of the conic sections except the circle. Referring to this curve Cantor writes:

"The problem of trisecting the angle led to the discovery of the first curve, aside from the circle, which is characterized by definite properties and whose development [generation] can be followed completely."

The three great geometrical problems of antiquity were:

1. To construct a rectangle whose area shall be equal to that of a given circle, that is, *squaring the circle.*

2. To divide an angle into three equal parts, that is, *trisecting the angle.*

3. To construct the edge of a cube whose volume shall be twice that of a given cube, that is, *duplicating the cube.*

By the time of Plato it came to be definitely required that "geometric constructions" should be made by means of ruler and compasses. It is now well known that under this restriction the solution of each of these problems is impossible. However, by avoiding this restriction at some point or other all these problems were solved. It is illuminating to note that with this restriction, the parabola, the ellipse, and the hyperbola are impossible of construction.

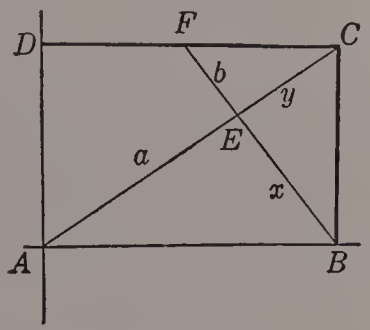
As was pointed out on pages 192, 193, the quadratrix may be used to square the circle and also to trisect the angle. Cantor concludes from an examination of the available evidence that "By 420 B.C. [and probably earlier, see quotation from Cantor above] Hippias of Elis had discovered a curve which could be used to trisect an angle and to construct a rectangle whose area is that of a given circle. From this last application the curve received [later] the name quadratrix." Cantor thinks it likely however, that the original purpose of this curve was

to trisect the angle. Certain it is that it is admirably adapted to this purpose.

The generation of the quadratrix is an example of what may be called "geometry of motion," which in spite of the restrictions laid down for geometrical construction was used quite extensively by the Greeks. It is supposed that Plato was largely responsible for the restriction on geometrical construction, but still the following method for duplicating the cube is attributed to him. In describing his construction we use modern notation.

In the figure, $ABCD$ is a rectangle and BF is perpendicular to AC.

Let $AE = a$, $BE = x$, $CE = y$, and $EF = b$.



Then from a well-known theorem in geometry, $a : x = x : y$ and $x : y = y : b$, and hence $x^2 = ay$ and $y^2 = xb$, whence

$$x^3 = a^2b.$$

If b is taken equal to $2a$, then $x^3 = 2a^3$.

If the lines BC and CD *are moved*, $ABCD$ remaining a rectangle, until they get into a position which makes $b = 2a$, then the line x is the edge of the cube whose volume is twice that of the cube whose edge is a. Plato is supposed to have invented a mechanical device for making this construction. Hence the problem was solved. Quite true, but not by geometric construction as required by Plato.

This problem of duplicating the cube was called the Delian problem. There are many stories of its origin. One is that a cubical altar having been made for the temple at Delos the god in wrath commanded that one with exactly twice its volume be made, "nor fail in the exact dimension." Hippocrates of Chios, in the second half of the fifth century before

Christ, showed that the problem is equivalent to finding two proportionals between two given lines. That is, in modern language, given a and b, find x and y so that

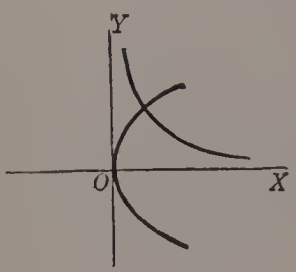
$$a : x = x : y = y : b \text{ which gives } y^3 = ab^2.$$

This led Menæchmus, a contemporary of Plato, to solve the problem by constructing two parabolas. Using modern notation and processes, let a be the edge of the given cube. Consider the two parabolas,

$$x^2 = 2ay \text{ and } y^2 = ax.$$

Then $x = \dfrac{y^2}{a}$, $x^2 = \dfrac{y^4}{a^2} = 2ay$, and $y^3 = 2a^3$.

Hence constructing these parabolas and finding the ordinate of the intersection point gives a line segment which is the edge of a cube whose volume is twice that of the given cube. Menæchmus also solved the Delian problem by making use of a parabola and a hyperbola:



Consider the curves $y^2 = px$ (1) and $xy = 2p^2$ (2), where p is any given constant.

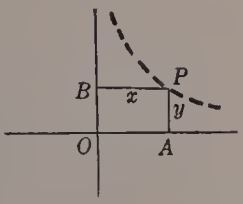
From (1) $x = \dfrac{y^2}{p}$. Substituting in (2)

$$y^3 = 2p^3,$$

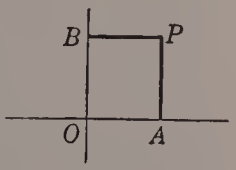
and the problem is again solved; but not by "geometric construction," for as noted above neither of these curves can be constructed in that way.

Thus we see that while the efforts of the Greeks failed to solve these problems by "construction," they did lead to the discovery and study of the curves which we now call conic sections. But the Greeks had no system of coördinates and did

not use equations of curves as we do. How did they define the parabola and the hyperbola?



Consider the equilateral hyperbola: Two lines, our coördinate axes, were drawn, say at right angles. Construct a rectangle $OAPB$ with area k. The locus of the point P is the hyperbola. The relation between the variables was usually written $BP : 1 = k : y$, k being given as a line segment.



The parabola was defined as follows: Construct axes as in the case of the hyperbola. For any segment BP take AP so that $BP : AP = AP : k$, k being a given line segment. In our language this reduces to $\overline{AP}^2 = k \cdot BP$.

Let us not be deceived into thinking that this was essentially our system of coördinates. The lines constructed for reference, OA and OB, were immediately related to each curve. There was no idea of using the same axes for different curves such as circles with different centers. A straight line was not thus referred to axes. There was, of course, no equation of a curve in our sense. The treatment was entirely in the spirit of Euclidean geometry. Proportions, such as are obtained in working with similar triangles, were used where we use equations.

The names parabola, ellipse, and hyperbola were first used by Apollonius, (see page 288). Certain facts about these curves were known. Thus Menæchmus knew about the asymptotes to the hyperbola.

Euclid (300 B.C.) wrote on conic sections but this work is lost and we know about it only from references by later Greek writers; our knowledge about the definite extent of this work is slight. Certainly, Euclid did not know that curves having the properties ascribed to these plane curves could be obtained as sections of a conical surface.

Archimedes (287–212 B.C.) found the area enclosed by a

parabola and a chord perpendicular to the axis, and also the area of the ellipse.

The great work of antiquity on conic sections is due to Apollonius of Perga (about 200 B.C.). In the century intervening between Euclid and Apollonius it had been discovered that the curves which we now call conic sections could be obtained by making plane sections of a cone. The sections of the conical surface were obtained by means of planes at right angles to an element of the cone, the kind of curve depending upon the angle of the cone. If the angle at the apex of the cone is acute the section is an ellipse, if this angle is a right angle the section is a parabola, and if obtuse, the section is a hyperbola. Apollonius obtained all conic sections from one conical surface by using different planes, the kind of curve depending upon the angle at which the plane cuts the cone. The names ellipse, parabola, and hyperbola were first used by him.

Apollonius, using Euclidean methods, obtained nearly all the properties of conic sections that are given in this book, and many others. Cantor enumerates among others the following: Diameters and conjugate diameters were defined and their existence proved, and it was shown that a conic has only one pair of conjugate diameters which are perpendicular to each other. The major and minor axes of the ellipse and hyperbola were defined "though the length of the minor axis for the hyperbola is not clear." The two vertices and the two foci of the ellipse and the hyperbola were discovered. Foci were defined as follows: A focus is a point F on the major axis $A'A$ such that $A'F \cdot FA = b^2$ (using modern terminology). The following proposition, for instance, was proved. "If a tangent to a conic meets the tangents at A' and A in P' and P then the circle described on $P'P$ as a diameter passes through the foci." The theorem on angles formed by focal radii with tangents and normals (see §§77, 91) were known to him, as were also the directions of tangents at the extremities of conjugate diameters

(see §139) and the theorems on the sum of the squares of conjugate diameters in the ellipse (see §144) and the difference of their squares in the hyperbola (see §149). He proved that in the ellipse the sum of focal radii drawn to a point is constant, while in the hyperbola the difference of these radii is constant. The center of central conics was found and the symmetry of these curves with respect to this point was established. He found methods for constructing a tangent to a conic through any point in the plane from which tangents can be drawn and he found how many normals to a conic can be drawn from a point.

Asymptotes were defined as follows: On a tangent to a hyperbola at P lay off PQ equal to the diameter parallel to PQ. Connect Q with the center O of the hyperbola. Then OQ is an asymptote.

Theorems like the following were stated and proved: The line connecting the intersection point of two tangents with the middle point of their chord of contact is a diameter. The triangle formed by the asymptotes and a tangent is of constant area.

If from a point on a parabola a normal to the curve is drawn and also a normal to the axis, then these meet the axis in points which are a constant distance apart. That is, the subnormal in the parabola is constant.

He proved that two distinct conic sections meet in four distinct points, in two points and are tangent to each other at one point, in two points, are tangent at one point, or do not meet at all.

Problems of much greater difficulty were stated and solved. One of the most far-reaching of these is:

From a given point find the length of the shortest and the longest line that can be drawn to a given conic.

The problem was to construct, on a given line, segments equal to these distances. This problem gives rise to certain

theories about curves which are now always treated by means of the calculus.

The so-called "garden method" (see §74) for constructing the ellipse, though an immediate corollary of a theorem of Apollonius, is of Arabic origin.

The focus of the parabola was not known to Apollonius; this was discovered five hundred years later by Pappus, who also was the first to use the general ratio definition of the conic, the kind of conic depending upon the value of this ratio, the eccentricity (see pp. 73, 85, 103). The pole and polar theory was not developed by Apollonius, though he stated and proved the following theorem.

If through an external point P of a conic a secant meeting the curve in R and S and two tangents are drawn, then the chord of contact meets the secant in a point Q such that the four points $PQRS$ form a harmonic ratio. That is, $PR : PS = QR : QS$.

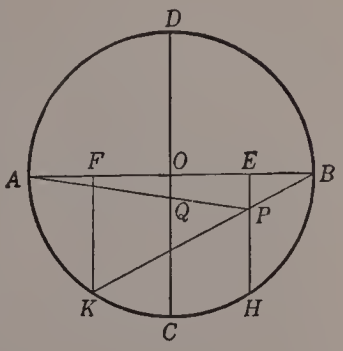
A problem proposed by Pappus but not solved by him is the following:

Given any number of straight lines in a plane, find the locus of a point so that the product of the distances from this point to any number of the lines shall be in a constant ratio to the product of the remaining distances.

This problem engaged the attention of Descartes and Newton (see page 293). With our present method the equation of the locus may be written down immediately by any fairly good beginner in analytic geometry.

We now turn to the story of the invention and use of some of the higher plane curves due to the Greeks. As noted above the earliest of these was the quadratrix whose invention is attributed to Hippias (about 420 B.C.) by whom it is supposed to have been used to square the circle and also to trisect an angle. The proof given on page 192 is attributed to Dinostratus, a brother of Menæchmus (see page 286).

The cissoid of Diocles was invented, as indicated by the name, by one Diocles, who lived probably in the second century before our era. His definition and method of construction was different from that given in §168, which is the usual modern method. Following is the construction and proof given by Diocles. In a circle draw two perpendicular diameters AB and CD meeting in O. Let $OE = OF$. Draw EH and FK perpendicular to AB and draw BK, meeting EH in P. The cissoid is the locus of P.



Two proportionals may now be constructed between any two lines of lengths a and b (see page 286). On OC find Q so that $AO : OQ = a : b$. Draw the line AQ meeting the cissoid in P. Then $AE : EP = a : b$.

Let $k \cdot a = AE$, then $k \cdot b = EP$.

Since $AE : EH = BF : FK = \dfrac{BE}{EP}$,

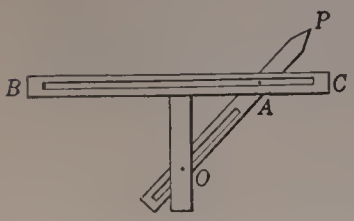
then $k \cdot a : EH = EH : EB = EB : k \cdot b$.

Dividing each term in the proportion by k,

$$a : \frac{EH}{k} = \frac{EH}{k} : \frac{EB}{k} = \frac{EB}{k} : b,$$

and the problem is solved.

The conchoid (§169) was invented by Nicomedes a contemporary of Diocles. It may be constructed mechanically by a device shown in the figure. Two rulers are fixed at right angles, with a third ruler OA having a fixed knob at A which slides along a horizontal slot in BC. In the vertical ruler is fixed a knob at O along which OA may slide. As the ruler



OA moves the distance AP remains fixed as does the point O. The point P describes the conchoid.

A system of coördinates was described and used long before analytic geometry was invented. We have seen that Apollonius and other Greeks described the location of a point by giving its distances and directions from two intersecting lines. To invent analytic geometry it was necessary to connect these two distances by means of an equation and thus to get an equation and its locus. "One of the most interesting medieval forerunners of analytic geometry was the work of Nicole Oresme (about 1323–1382). A teacher at Collège de Navarre, and later Bishop of Lisieux, this brilliant favorite of Charles the Fifth, spoke unsparingly but in finished form of the foibles and weaknesses of bishops and popes. An eminent stylist both in Latin and French, his writing exerted a wide influence even more than a hundred years after his death."

In Oresme's scheme a point was located by means of its "latitude" and "longitude." The latitude of a point was also called its ordinate, but our word abscissa was not used for another two hundred years. Oresme drew a semicircle with what we would call the origin as its center, and above our x-axis. The words positive and negative were not used in speaking of the longitude. In the semicircle the latitudes change rapidly, he said, near the left and the right ends of the figure; the rapidity of this change decreases as we approach the middle from both sides; at the middle point (the highest point of the semicircle) the latitudes do not change. In modern language we should speak of the derivative and its different values for different parts of the curve. On a straight line the ordinates change uniformly; that is they form an arithmetic progression for a set of equally spaced values of the longitude.

So wrote Oresme. But this is not analytic geometry; there is no use of equations to connect ordinates and longitudes. It is however, the stuff out of which analytic geometry can b.

made, though there is here but a slight element of advanee on the work of the Greeks.

We now turn to René Descartes (pronounced Dā-cart′) to whom the invention of analytic geometry is attributed historieally. A brief sketeh of his life seems justified. Born (1596) of a family belonging to the "demi-noblesse," he was assured of high soeial position and suffieient wealth to make him inde pendent for life. He attended a Jesuit school, but learned early to doubt the traditions which formed the basis of what his teachers had to tell him. Soeial life was not to his liking and very early he showed a decided bent for mathematics and for philosophical speculations. At the age of eighteen he concealed himself in Paris and for two years indulged his propensities undisturbed; then he was diseovered and dragged into soeial life. At twenty-one he went to the Netherlands and enlisted in the Army of Prinee Mauriee of Orange; later he went to Bavaria where he again enlisted in the army, traveled through Poland and the states of the present German republie, returned to France, and finally in 1629 went to Holland where he eame with a settled purpose to devote himself to his speculation, experimentation, and thinking. During the next twenty years he lived in seclusion, ehanging his abode thirteen times to avoid discovery. His eorrespondence was transmitted for him by a few confidential friends—chiefly Catholic priests. His fame had spread over Europe and in 1649 the twenty-three year old queen of Sweden learned about him from her friend the Freneh ambassador at her eourt. Christina, the daughter of the genius king Gustavus Adolphus, was a woman of remarkable ability of strength of will. Stoekholm was to be the Athens of the North. Seholars from all over Europe were invited to her court; after some hesitation Deseartes decided to join them and a vessel of the royal fleet was sent to wait upon him.

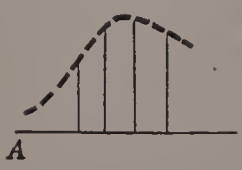
Deseartes was small of stature and never robust in health; he was in the habit of remaining in bed until noon, reflecting

and taking notes on a pad he always had with him. He was not equipped for the life that awaited him in the northern capital. His first duty was to draw up plans for a new academy of science about to be created. The young queen would dance until midnight and she wished to be instructed in philosophy at five in the morning. Descartes was to be made a full member of the Swedish nobility and an estate had already been selected for him. But

Es wäre zu schön gewesen
Es hätte nicht sollen sein.

The French ambassador, Chanut, the mutual friend of Christina and Descartes, was taken ill with pneumonia and Descartes attended him faithfully, until the same disease laid hold on him also. The ambassador recovered but Descartes died February 11, 1650.

In 1637 Descartes published his Géométrie. He proceeds as follows: Perpendiculars are erected to a straight line on which a definite initial point A is taken. The upper end-points of these perpendiculars lie on a curve. Thus two quantities are associated with each point, the length of the perpendicular and the distance from A to its foot.



If for every point of the curve these quantities satisfy the same equation then the curve is the locus of the equation.

Thus analytic geometry was born. The idea of two quantities determining a point was nearly two thousand years old, but connecting them by means of an algebraic equation was new. By this simple device it was made possible to investigate the properties of geometric figures by means of that powerful instrument, the equation. Note that Descartes does not construct the y-axis.

It is often said that analytic geometry consists in the application of algebra to geometry. As a matter of fact algebra

was used extensively in the solution of geometric problems long before the invention of analytic geometry. It is *the description of determining properties of a line or curve by means of an indeterminate equation in two variables* which constitutes plane analytic geometry. It is this which distinguishes the science with which we are dealing from all the geometry that existed before 1637.

Descartes did not make a systematic organization of analytic geometry. Very little of what is in this book is found in his work. Nowhere does he develop the equation even of a straight line.

To quote Cantor: "So far from starting with a description of the concepts involved in analytic geometry, such as we are accustomed to, Descartes nowhere makes specific statement of these; one is obliged to pick them out of his procedure as the work progresses."

After some preliminary observations and the solution of some second degree equations by means of ruler and compasses, he proceeds to the consideration of the problem of Pappus (see page 290). In its general form this problem is:

Given a set of straight lines, find the locus of a point P such that the product of the distances from P to some of these lines is equal to a given constant multiplied by the product of the distances from P to the remaining lines.

In briefly describing Descartes' method we will use our modern terminology. He first proved that if x, y are the "coordinates" of P, then the distance from P to a given line can be expressed in the form $Ax + By + C$ where A, B, C are constants depending upon the location of the line (see §40). The equation of the locus of P may now be written down at once. Suppose there are four lines and that the distances from P are given by $A_1x + B_1y + C_1$, $A_2x + B_2y + C_2$, $A_3x + B_3y + C_3$, $A_4x + B_4y + C_4$. Then the equation of the locus of P will be in one of the forms

$$(A_1x + B_1y + C_1)\ (A_2x + B_2y + C_2) = k(A_3x + B_3y + C_3)\ (A_4x + B_4y + C_4)$$

$$\text{or } A_1x + B_1y + C_1 = k(A_2x + B_2y + C_2)\ (A_3x + B_3y + C_3)\ (A_4x + B_4y + C_4).$$

That is, the equation will be of the second or the third degree depending upon the statement of the problem. If a larger number of lines is given the variety of resulting equations will of course be greater. This led at once to a classification of curves according to the degree of their equation which is mainly outside the field of this book. Descartes studied the construction of lines perpendicular to a curve (normals to the curve) and in particular does he show how to construct normals to what are known as Cartesian ovals. This class of curves is described in the Géométrie, but Descartes does not say how he came upon them. The folium of Descartes (see §176) is contained in a letter written in 1638.

Descartes also considered curves that do not lie in a plane. A general curve in space is projected upon two mutually perpendicular planes, and the curves in these planes determine the space curve completely. Since each of these projection-curves is determined by an equation, a space curve is determined by two equations each in two variables, and hence three variables in all (one variable may be common to the two equations). This is as far as Descartes goes in three dimensional analytic geometry. There is no equation of a surface.

Pierre de Fermat (1601–1665) discovered analytic geometry entirely independently and probably earlier than Descartes. "In important respects Fermat went far beyond Descartes" says Cantor. But Fermat failed to publish his work until after the appearance of Descartes' Géométrie and to Descartes therefore the historian assigns the discovery. Fermat gives the equation of a straight line through a given point on the x-axis. The equations of the circle, the parabola, ellipse, and hyperbola are all given, and the hyperbola is also referred to its asymptotes as axes.

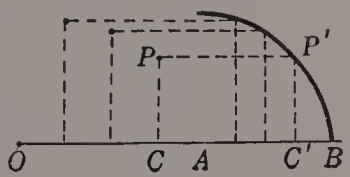
Descartes worked preferably in the first quadrant. That is, his coördinates were positive; in no case did he permit the independent variable to take negative values, though he did use negative ordinates. Newton was the first to use literal coefficients and permit these to take any real values, positive or negative; he worked with equal freedom in all four quadrants.

The theory of poles and polars was extended by Lahire (1640–1718) who proved again the theorem announced by Apollonius on the harmonic division of a secant by the chord of contact (see §133). Then he stated and proved the theorems: "If tangents meet on a line, the chords of contact meet in a point." The general theory (see chapter XI) came later and the words pole and polar were first used in 1810 and 1812 by Servois and Gergonne.

Space coördinates were first used systematically by Antoine Parent about 1700. He used the equation of the sphere for instance. Jacob Bernoulli used polar coördinates about 1691.

Some special items may be mentioned: Stevin (1583) constructed an ellipse with given semiaxes a and b, as follows.

Lay off $OA = a$ and extend so that $AB = b$. With AB as a radius construct a quadrant of a circle. Divide AB into a number of equal parts, four for instance, and divide OA into the same number of equal parts. At any division point C on OA construct an ordinate CP equal to the corresponding ordinate $C'P'$ on the circle. Then P is on the ellipse.



Mydorge proved that if from P, any point in the plane of a conic section, lines are drawn meeting the conic in Q and if R is a point on the line PQ such that the ratio $PQ : QR$ is constant, then the locus of R is a conic section similar to the first.

Maria Gaetana Agnesi of Milan (1718–1799) discovered the curve known as the "Witch" and sometimes as the "witch of Agnesi" (see §170).

Powers and Roots

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	51	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	60	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	70	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	80	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	90	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	100	10 000	1 000 000	10.000	4.642

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